On graphs whose spectral radius is bounded by $\frac{3}{2}\sqrt{2}$

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A preprint (to appear in Graphs and Combinatorics) with full details can be downloaded from

http://www.mat.univie.ac.at/~neum/papers.html#evmax

1 Known results

All graphs in this talk are connected, undirected, without loops or multiple edges.

Restrictions on the largest eigenvalue $\lambda_{\max}$ of graphs force these to have a very special structure.

- $\lambda_{\max} \leq 2$: **simply laced spherical and affine Dynkin diagrams.**
  Smith 1970

- $2 < \lambda_{\max} \le \sqrt{2+\sqrt{5}}$ (≈ 2.0582): **in addition certain trees with maximal degree 3 and at most two vertices of degree 3.**
  structurally restricted by Cvetkovic, Doob & Gutman 1982; **classification completed by** Brouwer & Neumaier 1989

- $\sqrt{2+\sqrt{5}} < \lambda_{\max}(G) \le \frac{3}{2}\sqrt{2}$ (≈ 2.1312): **certain quipus, resembling the knotted strings used by the Incas for information storage.**
  Renee Woo (now Los Alamos) 1993, but not written up until now

Cvetkovic, Doob, and Gutman used the following tools:

Hoffman's Lemma (HOFFMAN 1972)
Let e be an edge of G and H formed from G by deleting e and replacing it with a path of length two. Then

(i) $\lambda_{\max}(H) > \lambda_{\max}(G)$ if e is on a **pendant trail**, i.e., there exists a path $v_1, v_2, ..., v_k$ in G such that v_1 is an end point of e, v_k is univalent, and all other vertices in the path are of degree two.

(ii) Otherwise, $\lambda_{\max}(G) \geq \lambda_{\max}(H)$, with equality iff $G = C_n$ or $G = D_n$, where C_n is the n-gon, and D_n is the graph obtained from C_n by joining an extra vertex to a vertex of C_n.

From Perron-Frobenius theory;

Subgraph Lemma
If H is a (not necessarily induced) subgraph of G then $\lambda_{\max}(H) \leq \lambda_{\max}(G)$.

These two results are enough to eliminate all structurally wrong graphs by forbidden subgraph arguments.

To show that most of the structurally allowed graphs have the right largest eigenvalue, one needs an upper bound on these eigenvalues. Brouwer and Neumaier obtained these using exit values from NEUMAIER 1982.

A vector e (considered as a function on the vertices) is a **λ-partial eigenvector** with respect to the vertex $z \in G$ iff $e(z) = 1$, and the relation

$$\sum_{x \sim y} e(x) = \lambda e(y) \quad \text{for all } y \in G \setminus \{z\}$$

In this case, the number

$$\epsilon(z) = \lambda - \sum_{x \sim z} e(x)$$

is called the **λ-exit value** of G with respect to z.

Obviously, if the λ-exit value ϵ is zero, then λ is an eigenvalue of G, and e a corresponding eigenvector.

1.1 Theorem. *(*NEUMAIER 1982*)*

Let e be a λ-partial eigenvector of G with respect to $z \in G$ such that λ is not an eigenvalue of $G \setminus \{z\}$, $e > 0$, and $\epsilon(z) > 0$. Then $\lambda_{\max}(G) < \lambda$.

The same tools, but with a more complex analysis, are also sufficient to derive the new results.

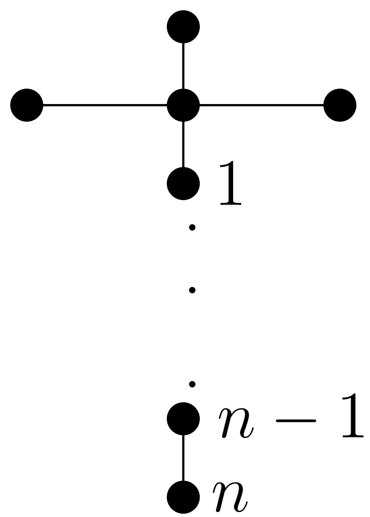
2 Restricting the structure

An open quipu is a tree G of maximum valency 3 such that all vertices of degree 3 lie on a path.

A closed quipu is a connected graph G of maximum valency 3 such that all vertices of degree 3 lie on a circuit, and no other circuit exists.

By Hoffman's Lemma, the largest eigenvalue decreases if we make the side chains (pendant trails) of a quipu shorter, or if we lengthen the paths between two branching knots (degree 3 vertices) of a quipu.

A dagger is a path with a 3-claw attached to an end vertex, i.e., one of the graphs $T_0(n), n \geq 2$.



$T_0(n)$

$\mathcal{S}$ denotes the family of graphs with largest eigenvalue > 2 and $\leq \frac{3}{2}\sqrt{2}$.

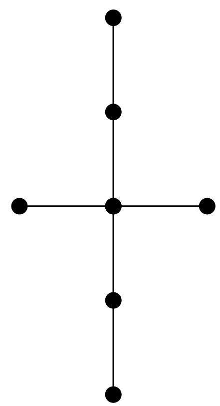
No G can attain $\lambda_{\max}(G) = \frac{3}{2}\sqrt{2}$, since this is not an algebraic integer.

2.1 Theorem. *A graph G in $\mathcal{S}$ is either an open quipu, a closed quipu, or a dagger.*

Conversely, every dagger and all quipus with (in a precise sense) sufficiently long gaps between the vertices of degree 3 belong to $\mathcal{S}$.

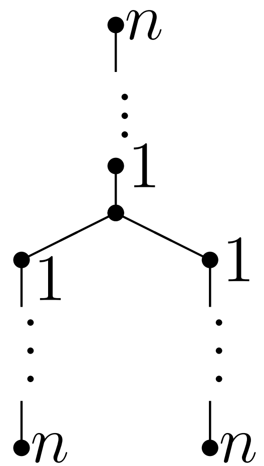
By the Subgraph Lemma and Smith's classification:

2.2 Proposition. *Any graph $G \in \mathcal{S}$ has maximum degree $d(G) \in \{3, 4\}$. Moreover, T_3 cannot be a subgraph of G.*



T_3

For the daggers $T_0(n)$ and the graphs $T_1(n)$,



$T_1(n), n \geq 1$

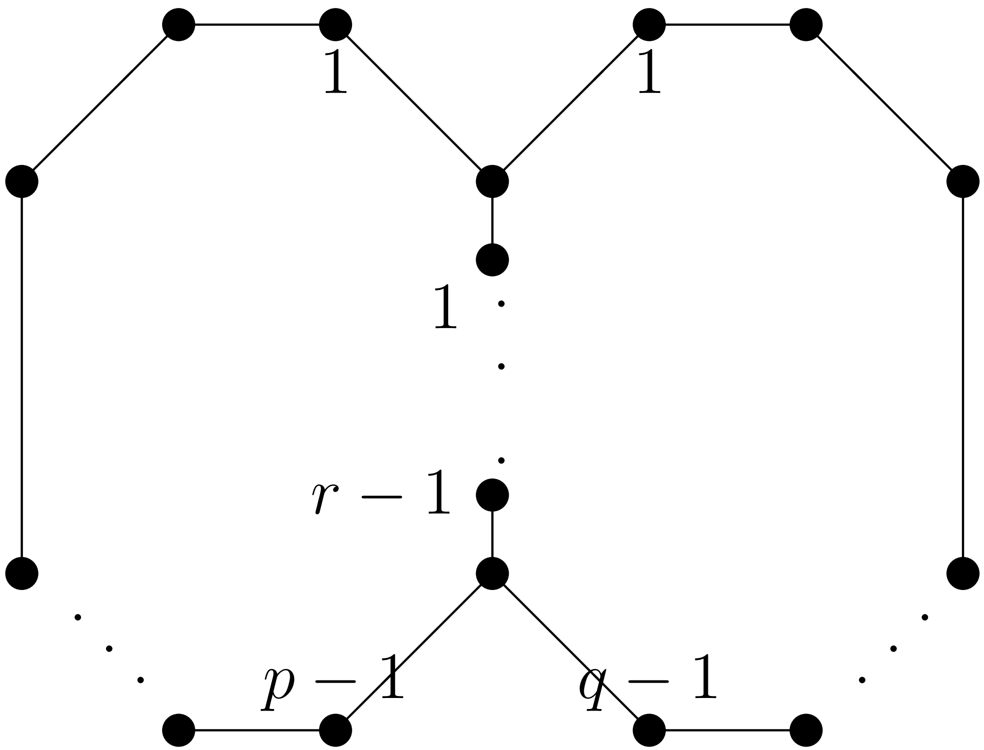
another result of Hofmann gives

$$\lim_{n\to\infty} \lambda_{\max}(T_0(n)) = \lim_{n\to\infty} \lambda_{\max}(T_1(n)) = \lambda_0.$$

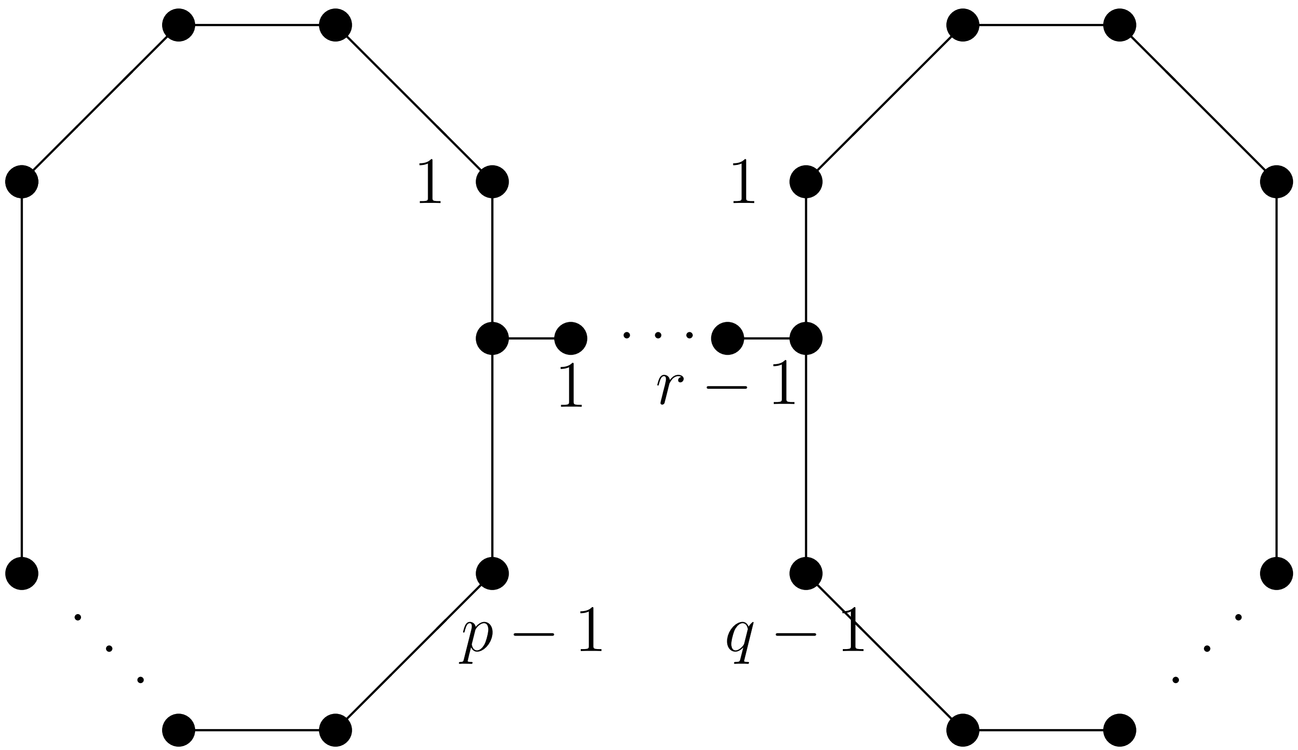
Together with the Subgraph Lemma and Hoffman's Lemma, this implies:

2.3 Proposition. *A graph $G \in \mathcal{S}$ is either a tree or has exactly one circuit.*

We only have to exclude graphs of the form $Z_1(p, q, r)$ or $Z_2(p, q, r)$ as subgraph, for $p, q \geq 2, r \geq 1$. (For example, two circuits with a single common vertex form a graph $Z_1(p, q, 1)$.)



$Z_1(p, q, r), p, q \geq 2, r \geq 1$

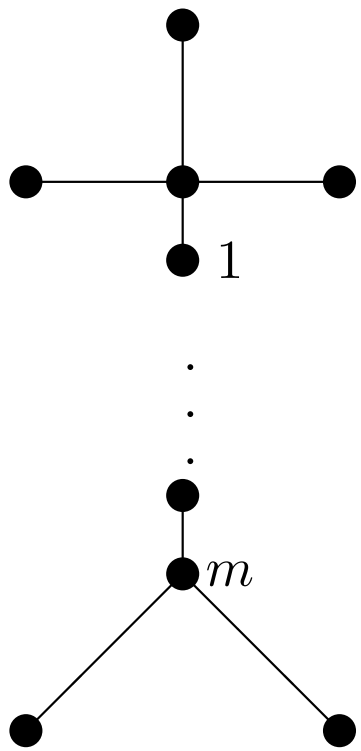


$Z_2(p,q,r), p,q \geq 3, r \geq 1$

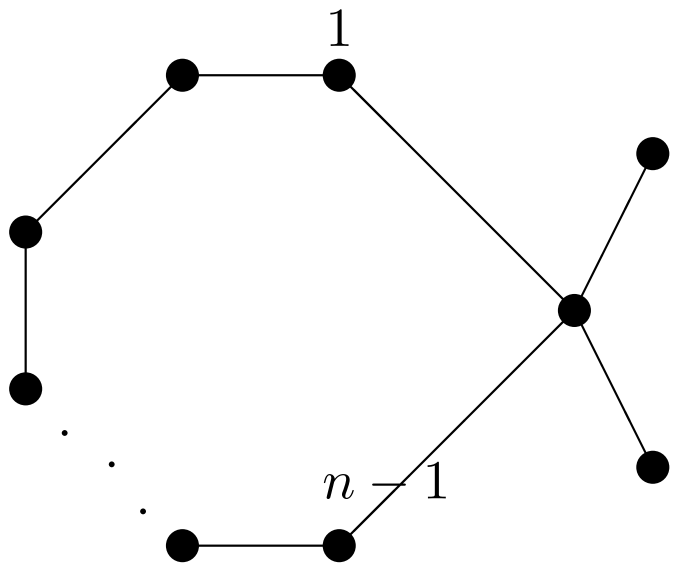
With the same tools and the above results, one now finds:

2.4 Proposition. *The set of graphs in $\mathcal{S}$ with $d(G) = 4$ is precisely the set $\{T_0(n) \mid n \geq 2\}$; i.e., a graph with $d(G) = 4$ is in $\mathcal{S}$ iff it is a dagger.*

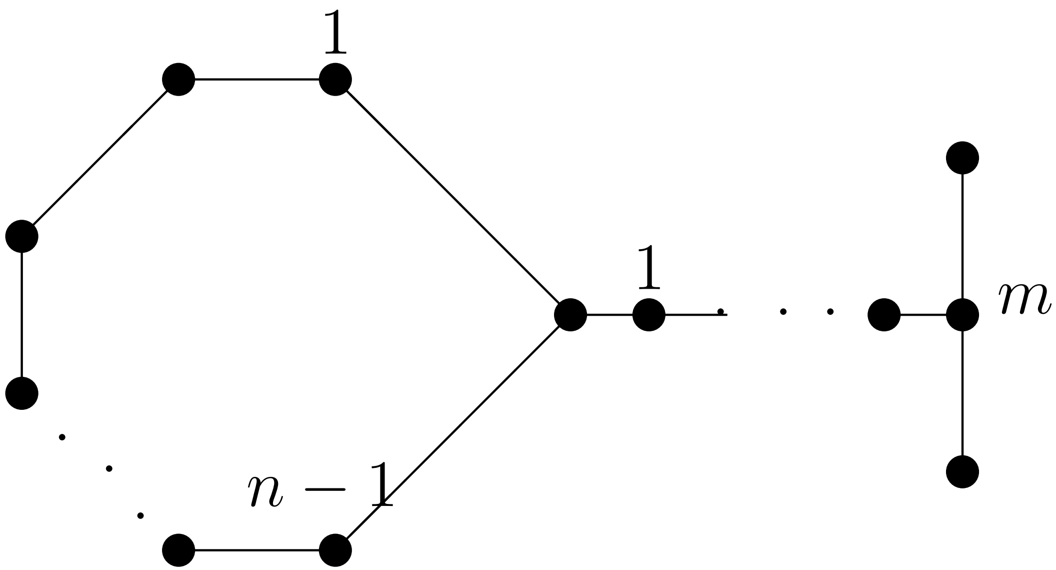
This requires excluding subgraphs $T_2(m)$, $Z_4(n)$ and $Z_3(m, n)$ for $m \geq 1, n \geq 3$.



$T_2(m), m \geq 1$



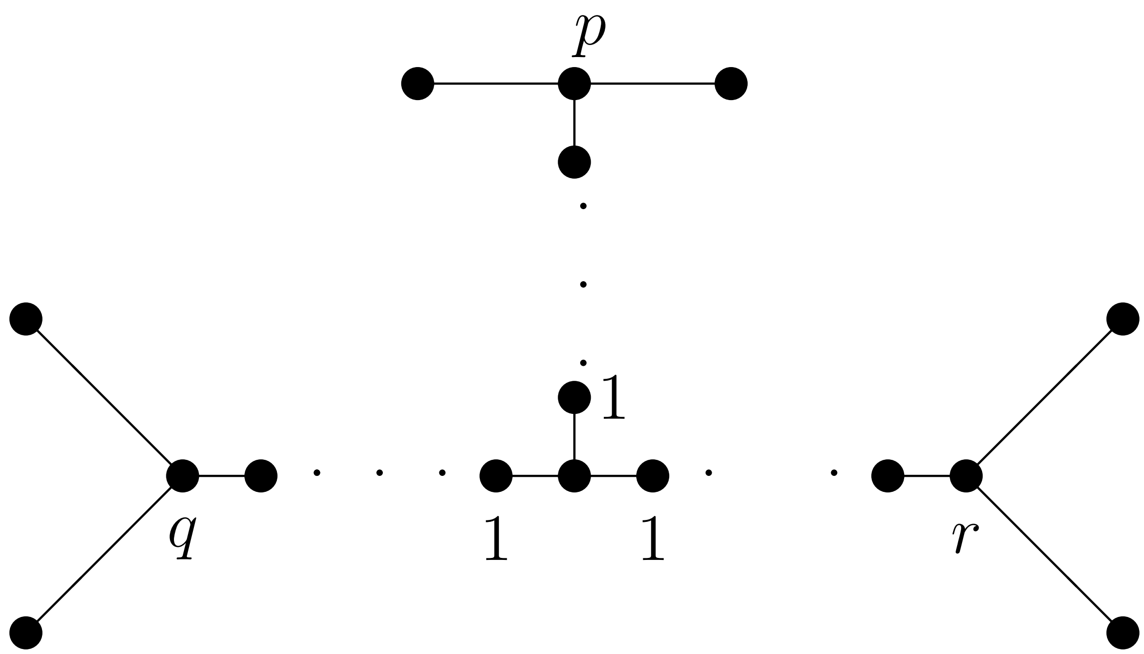
$Z_4(n), n \geq 3$



$Z_3(m, n), m \geq 1, n \geq 3$

2.5 Proposition. *If $d(G) = 3$, then all the vertices of degree 3 lie on a path. Furthermore, if G is not a tree, then all vertices of degree 3 lie on the circuit. Thus G is an open or closed quipu.*

This time, we need to exclude the subgraphs $T_4(p, q, r)$ for $p, q, r \geq 1$.

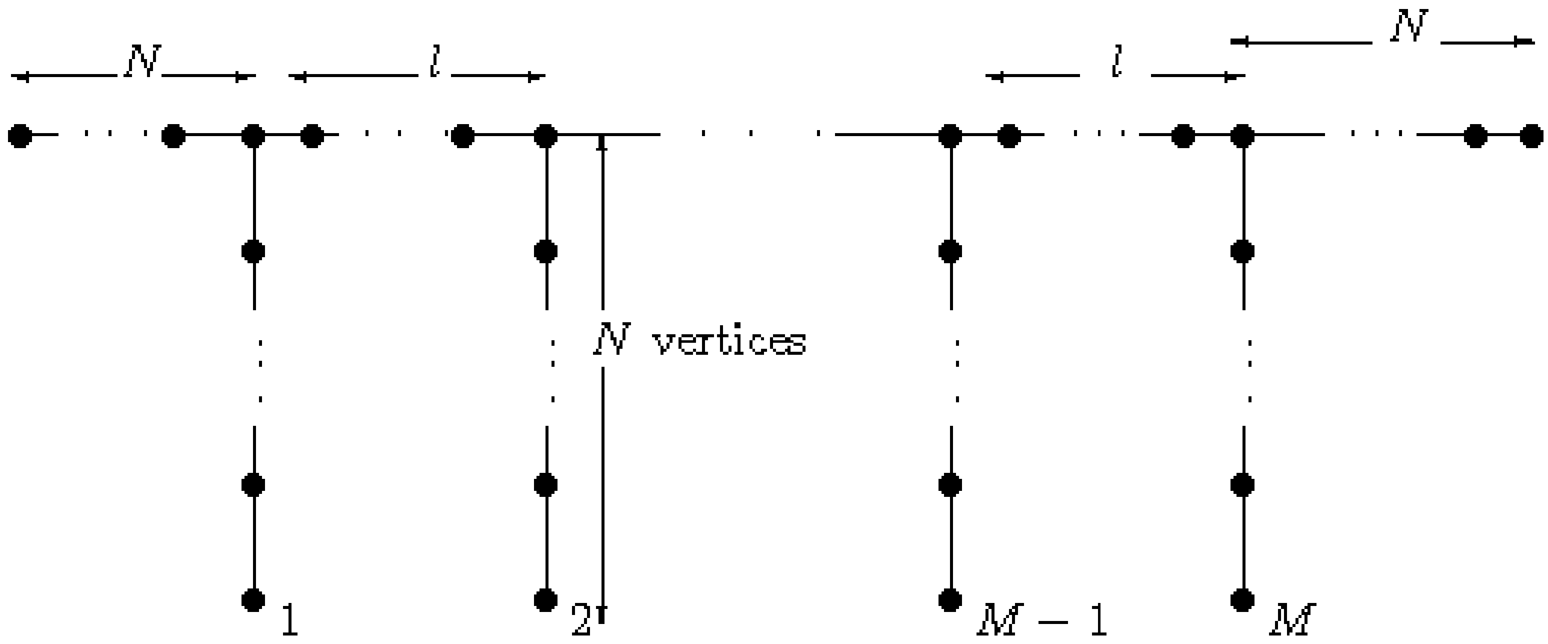


$T_4(p, q, r), p, q, r \geq 1$

3 Widely spaces quipus

Now we work towards the converse.

For fixed positive integers M, N and another integer $l > 1$, we define the graph $G_{M,N,l}$ as the uniform open quipu with M branch points connected by chains of length l, having $M + 2$ side chains of length $N - 1$, cf. the following diagram.



N
l
l
N
N vertices
1
2
M − 1
M

3.1 Theorem. *For every* $M, N, l \in \mathbb{N}^+$, *there exist a number* $L_{M,N,l}$ *such that* $l > \tilde{L}_{M,N,l}$ *implies* $G_{M,N,l} \in \mathcal{S}$.

The proof is based on the explicit computations of exit values, and requires the explicit evaluation of a number of finite sums, followed by limiting arguments to ensure the correct signs.

(The details are somewhat messy.)

The theorem implies that for every positive integer N, $\mathcal{S}$ contains open quipus with arbitrarily many side chains of length N.

Using Hoffman's Lemma, it is easy to relax the equal length restriction on gaps and side chains, resulting in a much larger family of open quipus in $\mathcal{S}$.

A similar argument handles the case of closed quipus, with analogous results.

The set of minimal forbidden subgraphs for $\mathcal{S}$ is infinite, as it contains (at least) all graphs $T_2(n)$ $(n \geq 1)$ defined below.

A result by SHEARER 1989 that every number larger than $\sqrt{2+\sqrt{5}}$ is the limit of the largest eigenvalues of a sequence of open quipus called **caterpillars**.

This suggests that a complete description of the graphs in $\mathcal{S}$ should be rather intricate.

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