Dual polar spaces as extremal distance-regular graphs

Arnold Neumaier

Institut für Mathematik, Universität Wien
Strudlhofgasse 4, A-1090 Wien, Austria
email: Arnold.Neumaier@univie.ac.at
WWW: http://www.mat.univie.ac.at/~neum/

Dedicated to the Memory of Jaap Seidel

Abstract. A new inequality for the parameters of distance-regular graphs is proved. It implies that if there are infinitely many distance-regular graphs with fixed λ, μ, a_i and c_i containing an induced quadrangle then necessarily $c_{i+1} \geq 1 + (\mu - 1)c_i$. As the dual polar graphs show, this inequality is best possible. Some related results are also discussed.

1 A new inequality

The family of distance-regular graphs coming from dual polar spaces (see Brouwer et al. [1] for notation and terminology) has parameters of the form

$$c_i = \frac{q^i - 1}{q - 1}, \quad a_i = \lambda c_i, \quad b_i = k - a_i - c_i, \tag{1}$$

and for each prime power q and $\lambda = q^e(q+1), e \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$, there is an example with arbitrarily large values of k (and large diameter). One may therefore ask about partial lists of intersection numbers realized by infinitely many distance regular graphs. The purpose of this note is to show that the dual polar parameter sets are extremal with respect to this point of view.

1.1 Theorem. *The parameters of a distance-regular graph satisfy the restriction*

$$\begin{aligned} &[c_i + p^i_{2i-1}, c_i a_{i-1}] + [a_i + p^i_{2i},\ c_i(b_{i-1} - 1)] \\ &\leq c_i(c_i - 1)\max(\lambda - 1, \mu - 1) - c_i c_{i-1}(\mu - 1) \end{aligned} \tag{2}$$

for $i = 2, \ldots, d$, *where*

$$p^i_{2i} = \mu^{-1}(c_i b_{i-1} + a_i(a_i - \lambda) + b_i c_{i+1} - k)),$$

$$p^i_{2i-1} = \mu^{-1} c_i (a_i + a_{i-1} - \lambda),$$

$$[p,q] = \begin{cases} \left\lfloor \frac{q}{p} \right\rfloor \left(2q - p - \left\lfloor \frac{q}{p} \right\rfloor p\right) & \textit{if } p \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Fix α, β at distance i. For $j \in \{i, i-1\}$, let $S_j := \Gamma_j(\alpha) \cap (\Gamma(\beta) \cup \Gamma_2(\beta))$, and denote by $N(\gamma)$ the number of points in $\Delta := \Gamma_{i-1}(\alpha) \cap \Gamma(\beta) \cap \Gamma(\gamma)$. Counting in two ways the number of configurations involving α, β, γ and $0, 1$, or 2 points of Δ gives

$$\sum_{\gamma \in S_j} 1 = p^i_{1j} + p^i_{2j}, \tag{3}$$

$$\sum_{\gamma \in S_j} N(\gamma) = c_i (p^{i-1}_{1j} - \delta_{ij}), \tag{4}$$

$$\sum_{\gamma \in S_{i-1} \cup S_i} N(\gamma)(N(\gamma) - 1) \leq c_i((c_i - 1)\max(\lambda - 1, \mu - 1) - c_{i-1}(\mu - 1)). \tag{5}$$

Since

$$N(N-1) = s(2N - 1 - s) + (N - s)(N - s - 1) \geq s(2N - 1 - s)$$

for any integer s we can bound the left hand side of (5) from below by

$$\sum_{j=i-1}^{i} s_j \sum_{\gamma \in S_j} (2N(\gamma) - 1 - s_j) = \sum_{j=i-1}^{i} s_j (2c_i (p^{i-1}_{1j} - \delta_{ij}) - (1 + s_j)(p^i_{1j} + p^i_{2j})). \tag{6}$$

Choosing the integers s_j to maximize this lower bound results in the inequality (2). □

A form of the special case $i = 2$ of the above bound with slightly weaker assumptions – indeed, the above proof only requires 'distance regularity up to some distance' – is in Brouwer et al. [1, Proposition 5.8.1].

1.2 Corollary. *If*

$$k > a_1^2 + \mu(a_i + c_i) + \mu \frac{c_i(c_i - 1)}{2} \max(\lambda - 1, \mu - 1) \tag{7}$$

then

$$c_{i+1} \geq 1 + (\mu - 1) c_i. \tag{8}$$

Proof. For $s_{i-1} = 0,\ s_i = 1$, the lower bound (6) for (5) takes the value

$$2(c_i(b_{i-1}-1)-a_i-p^i_{2i}) = \frac{2}{\mu}(k-a_i(a_i-\lambda)-\mu(a_i+c_i)-b_ic_{i+1}+c_i(\mu-1)b_{i-1}). \tag{9}$$

If (8) is violated then $c_{i+1} \le (\mu-1)c_i$, hence

$$-b_ic_{i+1} + c_i(\mu-1)b_{i-1} \ge c_i(\mu-1)(b_{i-1}-b_i) \ge 0. \tag{10}$$

Thus (6) is bounded from below by $\frac{2}{\mu}(k - a_i(a_i-\lambda) - \mu(a_i+c_i))$, and from above by $c_i(c_i-1)\max(\lambda-1, \mu-1)$, contradicting (6). Therefore (8) must hold. □

1.3 Corollary. *If there are infinitely many distance-regular graphs with fixed λ, μ , a_i, and c_i containing an induced quadrangle then necessarily $c_{i+1} \ge 1 + (\mu-1)c_i$. Equality holds, e.g., for dual polar graphs.*

Proof. (i) If a distance regular graph contains an induced quadrangle then $d \le k$ by the diameter bound of Terwilliger [4] (cf. [1, Corollary 5.2.2]). Now if (8) is violated then k (and therefore d) is bounded by the right hand side of (6). Hence the number of vertices is bounded, too. Thus there are only finitely many possibilities, contradition.

(ii) For dual polar graphs, (1) holds. Hence $\mu = q+1$ and $c_{i+1} = 1+(\mu-1)c_i$, so that (8) holds with equality. □

Note that graphs violating (8) satisfy $\mu > 1$. Since there are only three known quadrangle-free distance regular graph with $\mu > 1$ [1, p. 36], the hypothesis about quadrangles is almost certainly superfluous.

It would be interesting if the arguments used in the proofs of the above results could be tightened in such a way that it gives useful geometric information about graphs with $c_{i+1} = 1 + (\mu-1)c_i$ and large k. Ideally, one would like to conclude that there are no induced subgraphs $K_{1,1,2}$ (a 4-clique minus an edge); then the standard characterization theorems for Hamming graphs and dual polar graphs (cf. [1, Chapter 9]) apply (unless $\lambda = 0$ or $\mu = 1$).

The assumption in (7) is convenient but can be weakened by using the upper bound from (5) and the lower bound from (10) together with $b_{i-1} - b_i = c_i - c_{i-1} + a_i - a_{i-1}$. In general, this gives a messy expression. However, in the bipartite case, where all $a_i = 0$, it reproduces the following result of Caughman [3].

1.4 Corollary. *For a bipartite distance-regular graph,*

$$k > c_i + \binom{\mu - 1}{2} c_i(c_i - c_{i-1} - 1) \quad \Rightarrow \quad c_{i+1} \geq 1 + (\mu - 1)c_i.$$

2 Distance-regular graphs with fixed λ, μ

It is likely that much stronger results than Corollary 1.3 hold, but current techniques do not seem to give a handle for attack. The final goal would be the following conjecture.

Conjecture. For given λ and μ, there are only finitely many distance-regular graphs of diameter > 3 that contain an induced subgraph $K_{1,1,2}$. Any such graph has diameter $d \leq \lambda/2 + 1$.

Note that the Johnson graphs $J(2d, d)$ have equality in this conjectured diameter bound.

In the remainder of this section, we collect some observations in support of the conjecture. Clearly, the conjecture holds trivially if $\lambda = 0$ or $\mu = 1$. For $\lambda > 0, \mu > 1$, all but finitely many of the known distance regular graphs with diameter $d > 3$ have parameters of the form either (*"classical parameters"*)

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right), \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right), \tag{11}$$

with

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{cases} i & \text{if } q = 1, \\ (q^i - 1)/(q - 1) & \text{otherwise} \end{cases}$$

for some integer $q \neq -1, 0$, or (*"partition parameters"*)

$$b_i = (m - i)(1 + \alpha(m - 1 - i)), \quad c_i = i(1 + \alpha(i - 1)) \quad \text{for } i < d, \tag{12}$$

$$b_d = 0, \quad c_d = \gamma d(1 + \alpha(d - 1)), \quad (m, \gamma) \in \{2d, 2), (2d + 1, 1)\} \tag{13}$$

(see Chapter 6 of [1]; α, β are *not* necessarily integers).

If $\alpha = 0$, the known realizations of these parameter sets with $d > 3$ are near polygons (Hamming graphs, dual polar spaces, and folded cubes), which have no $K_{1,1,2}$. We investigate the case $\alpha \neq 0$.

2.1 Proposition. *A distance-regular graph with partition parameters* (12), (13) *and* $\alpha \neq 0$ *has diameter* $d \leq \frac{1}{4}(\lambda+2)$, *valency* $k \leq \frac{1}{4}(\lambda+2)^2$, *and* $\mu \geq 4$. *In particular, given* λ, *there are only finitely many possibilities.*

Proof. We have

$$k = b_0 = m + \alpha m(m-1), \quad \lambda = k - 1 - b_1 = 2\alpha(m-1), \quad \mu = 2 + 2\alpha. \quad (14)$$

The cases $\mu \in \{1, 3\}$ are impossible by Bussemaker & Neumaier [2, Theorem 3.3], and $\mu \neq 2$ since $\alpha \neq 0$. Hence $\mu \geq 4$, $\alpha \geq 1$, $m \leq 1 + \lambda/2$, $d \leq m/2 \leq (\lambda+2)/4$, and $k = m(1 + \lambda/2) \leq (\lambda+2)^2/4$. □

The realizations of partition parameters are classified in [2], apart from finitely many undecided cases.

2.2 Proposition. *A distance regular graph of diameter* $d \geq 3$ *with classical parameters* (11) *has*

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} + \kappa \begin{bmatrix} i \\ 2 \end{bmatrix}, \quad a_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(\lambda - \kappa \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} i \\ 1 \end{bmatrix} \lambda - \kappa(q+1) \begin{bmatrix} i \\ 2 \end{bmatrix}, \quad (15)$$

$$k = a_d + c_d = \begin{bmatrix} d \\ 1 \end{bmatrix} (\lambda + 1) - \kappa q \begin{bmatrix} d \\ 2 \end{bmatrix} \quad (16)$$

for some integer $\kappa \geq 0$, *where*

$$\begin{bmatrix} i \\ 2 \end{bmatrix} = \begin{cases} i(i-1)/2 & \text{if } q = 1, \\ (q^i - 1)(q^{i-1} - 1)/(q^2-1)(q-1) & \text{otherwise.} \end{cases}$$

Proof. $\kappa := \alpha(q+1) = \mu - 1 - q$ is an integer, and the formulas follow directly from $b_d = 0$, $k = b_0$, $\lambda = k - 1 - b_1$. Since $0 < c_3 = (q^2+q+1)(1+\kappa)$, we have $\kappa > -1$, hence κ is nonnegative. □

Note that $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} i \\ 2 \end{bmatrix}$ are integers, and $\begin{bmatrix} i \\ 2 \end{bmatrix} > 0$ for $i > 1$. We have $\kappa = 0$ iff $\alpha = 0$, and $\kappa > 0$ otherwise.

2.3 Proposition. *If a distance-regular graph with classical parameters contains an induced subgraph* $K_{1,1,2}$ *then either* $\kappa = 0$ *or*

$$0 < q \leq \mu - 2, \quad d \leq \lambda/2 + 1.$$

In particular, given λ *and* μ, *there are only finitely many possibilities with* $\kappa \neq 0$.

Proof. Suppose that $\kappa \neq 0$. Then $\kappa > 0$ and $q = \mu - 1 - \kappa \leq \mu - 2$. Moreover,

$$\lambda \geq \kappa \begin{bmatrix} d-1 \\ 1 \end{bmatrix}, \tag{17}$$

since $0 \leq a_d = \left[{d \atop 1}\right](\lambda - \kappa\left[{d-1 \atop 1}\right])$. If $q < 0$ then Terwilliger [5] proved the absence of subgraphs $K_{1,1,2}$; hence $q > 0$.

If $q = 1$, the graphs are classified [1, Theorem 6.1.1], giving $\kappa = 2\alpha \in \{0, 2, 4, 8\}$; hence $\lambda \geq 2d - 2$ by (17). If $q > 1$, $d \geq 3$ then (17) implies $\lambda \geq 2d - 2$ except when $d = 3$, $q = 1$, $\kappa = 1$, $\lambda = 3$. This would correspond to an intersection array $\{14, 10, 4; 1, 4, 14\}$, which is not feasible. Finally, if $q > 1$, $d \leq 2$ then $\lambda \geq 2d - 2$ holds since the existence of a $K_{1,1,2}$ implies $\lambda \geq 2$.

Hence $\lambda \geq 2d - 2$ holds generally, giving $d \leq \lambda/2 + 1$. □

As Table 6.6 of [1] shows, there are several infinite families of regular near polygons with fixed λ, μ (and indeed with fixed $a_1, \ldots a_s, c_1, \ldots, c_s$), among then the known graphs with classical parameters and $\kappa = 0$. The graphs with classical parameters and $\kappa = 0$ containing no $K_{1,1,2}$ must be regular near polygons [1, Theorem 6.4.1], and are completely classified for $\lambda > 0$ ([1, Theorem 9.4.4]).

The graphs with classical parameters and $q = -r < 0$ also contain such infinite families. For $d > 3, \lambda > 0, \mu > 1$, they have been nearly classified by Weng [6], who showed that in this case

$$\kappa = s(r+1),\ \ \lambda = s - 1,\ \ \mu = \kappa + 1 - r$$

for some prime power r and some positive integer

$$s \in \{r, r-1, (r+1)/2\}.$$

The first two cases are realized uniquely by dual polar graphs ${}^2A_{2d-1}(r)$ and Hermitian forms graphs $\mathrm{Her}_r(d)$, while the third case is apparently open. For $\lambda = 0$ or $\mu = 1$ little is known, but Weng [7] showed that for $d > 3,\ \mu = 1,\ 1 < \lambda < r - 1$ is impossible.

For $d \leq 3$, the situation seems to be different. I haven't looked at the diameter three case, but for strongly regular graphs ($d = 2$) one finds the following.

2.4 Proposition. *If there are infinitely many connected strongly regular graphs with fixed* λ, μ *then one of the following three cases holds:*

(i) $\lambda = \mu - 2$,

(ii) $\mu = t^2,\ \lambda = t^2 \pm 2t$ *for some* $t > 0$,

(iii) $\mu = t^2 \pm t,\ \lambda = t^2 \pm 3t$ *for some* $t > 0$.

Proof. If we have a conference graph then $v = 4\mu + 1$ is fixed, so that there are only finitely many choices. Otherwise, cf. [1, Theorem 1.3.1], the eigenvalues r, s are integral, and the parameters are determined by μ, r, s as

$$k = \mu - rs,\ \lambda = \mu + r + s.$$

Then

$$(s+1)k(k-s) \equiv 0 \bmod n := r - s,$$

since the multiplicity f must be an integer. It follows that

$$\lambda - \mu \equiv 2r \equiv 2s \bmod n,\ (\lambda - \mu)^2 \equiv 4(\mu - k) \bmod n,$$

and hence

$$(\lambda - \mu + 2)(4\mu - (\lambda - \mu)^2)(4\mu - (\lambda - \mu)^2 - 2(\lambda - \mu)) \equiv (2s+2)4k(4k - 4s) \equiv 0 \bmod n.$$

If the left hand side is nonzero, this leaves only finitely many choices for n, hence of $r = \frac{1}{2}(\lambda - \mu + n)$ and $s = \frac{1}{2}(\lambda - \mu - n)$, and hence only finitely many graphs. If the left hand side vanishes, one of the factors vanishes, giving the three cases (i)–(iii). □

Note that every connected strongly regular graph is classical in two ways, realized by any of the two settings

$$q = -1 - s > 0, \quad \kappa = \mu + s,$$

$$q = -1 - r < 0, \quad \kappa = \mu + r.$$

The case $\kappa = 0$ corresponds to $s = -\mu, r = \lambda$, which defines pseudo generalized quadrangles.

References

[1] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, Springer, Berlin 1989.

[2] F.C. Bussemaker and A. Neumaier, Exceptional graphs with smallest eigenvalue -2 and related problems, Math. Comput. 59 (1992), 583–608.

[3] J. Caughman, Intersection numbers of bipartite distance-regular graphs, Discrete Math. 163 (1997), 235–241.

[4] P. Terwilliger, Distance-regular graphs with girth 3 and 4, I, J. Combin. Th. (B) 39 (1985), 265–281.

[5] P. Terwilliger, Kite-free distance-regular graphs, Europ. J. Combinatorics 16 (1995), 405–414.

[6] C. Weng, Classical distance-regular graphs of negative type, J. Combin. Theory B 76 (1999), 93–116.

[7] C. Weng, D-bounded distance-regular graphs, Europ. J. Combinatorics 18 (1997), 211–229.