of C is the minimal integer t such that the spheres of radius t around the codewords of C cover all vertices of H.

The *subconstituents* of H w.r. to C are the sets

$$C_l := \{x \in H \mid d(x, C) = l\};$$

in particular,

$$C_0 = C, \qquad C_l \neq \emptyset \Leftrightarrow l \leq t_C.$$

A code C is called *completely regular* if, for all $l \geq 0$, every point $x \in C_l$ has the same number c_l of neighbours in C_{l-1} and the same number b_l of neighbours in C_{l+1}. A trivial counting argument gives the following.

Proposition 1.1. *C is completely regular iff there is a matrix L (indexed by $0, 1, \ldots, t = t_C$) such that*

$$|H(x) \cap C_j| = L_{lj} \quad \textit{for all } x \in C_l; \tag{1}$$

in this case, L is tridiagonal,

$$L = \begin{pmatrix} a_0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & b_{t-1} \\ & & & & c_t & a_t \end{pmatrix},$$

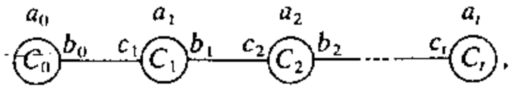
with

$$a_l = \bar{k} - b_l - c_l \geq 0, \qquad c_0 = b_t = 0. \tag{2}$$

The matrix L is called the *intersection matrix* of C, and the *intersection array*

$$i(C) := \{b_0, \ldots, b_{t-1}; c_1, \ldots, c_t\} \tag{3}$$

is a short way to specify the independent parameters. A useful geometric visualization is the *distribution diagram* of C:

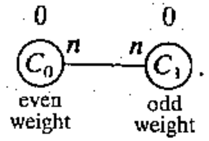


which indicates the number of edges between and within the subconstituents leaving a particular vertex.

Examples. A wealth of nontrivial examples is discussed in Brouwer et al. [2, Chapter 11]. The definition given there is different from ours, but Theorem 4.1

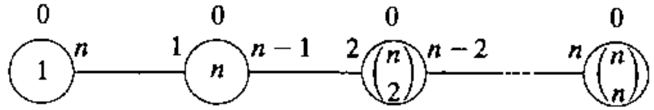
below implies the equivalence of the definitions for the case discussed there, where H is a distance regular graph. Here we only give some trivial examples illustrating the definitions given.

(1) In the n-cube, the set C of codewords of even weight (number of nonzeros) is completely regular with covering radius $t = 1$ and distribution diagram



(2) For any completely regular code C with covering radius t, the subconstituent C_t is also completely regular, with reversed intersection array and distribution diagram.

(3) In the n-cube, each *singleton code* $C = \{x\}$ is completely regular with covering radius $t = n$ and distribution diagram



A regular graph in which all singleton codes are completely regular is called *distance-regular*. It is easy to show that all singleton intersection matrices (arrays) must be identical, and define the *intersection matrix* (*array*) of H. In particular, the n-cube is distance-regular with intersection array

$$\{n, n-1, \ldots, 1; 1, 2, \ldots, n\}.$$

For theory and construction of distance-regular graphs see [1] (introductory) and [2] (comprehensive).

2. Spectral properties (Lloyd's theorem)

The *eigenvalues* of a completely regular code C are the eigenvalues of its intersection matrix L; they are distinct real numbers since L is tridiagonal and nonnegative. An important theorem due to Lloyd [4] (cf. van Lint [5, 6]) for the special case of perfect codes relates the eigenvalues of C to spectral properties of the graph H. To formulate Lloyd's theorem we define the *distribution matrices* A_i, indexed by pairs of vertices of H, with entries

$$(A_i)_{xy} := \begin{cases} 1 & \text{if } d(x, y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 1$ we obtain the *adjacency matrix* $A = A_1$ of H, whose eigenvalues are called the *eigenvalues* of H. If H is distance regular, the eigenvalues of H are precisely the eigenvalues of its intersection matrix, with a suitable multiplicity [1, 2]; e.g., the eigenvalue of the n-cube are $n - 2i$ $(i = 0, \ldots, n)$.

Theorem 2.1. *Every eigenvalue of a completely regular code in H is an eigenvalue of H.*

Proof. Define the *characteristic vectors* g_l of H with entries

$$(g_l)_x := \begin{cases} 1 & \text{if } x \in C_l, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(Ag_i)_x = \sum_y A_{xy}(g_i)_y = |H(x) \cap C_i| = L_{li} \quad \text{for } x \in C_l,$$

so that

$$Ag_i = \sum_l L_{li} g_l = c_{i+1}g_{i+1} + a_i g_i + b_{i-1}g_{i-1}. \tag{4}$$

Now suppose that $Lu = \theta u$, $u \neq 0$. Then

$$g := \sum u_i g_i \neq 0$$

satisfies

$$Ag = \sum_i u_i(Ag_i) = \sum_i u_i\Big(\sum_l L_{li} g_l\Big) = \sum_l (Lu)_l g_l = \sum_l \theta u_l g_l = \theta g.$$

Thus θ is an eigenvalue of A. □

More generally, Lloyds theorem holds with the same proof for *equitable partitions* $C_0, \ldots, C_t$ of H, where now L is allowed to be an arbitrary matrix with (1). See Godsil and McKay [3].

3. Combinatorial properties

The intersection array determines the size of a completely regular code C and its subconstituents. Indeed, writing $k_l := |C_l|$ for the size of C_l we have the following.

Proposition 3.1. *For a completely regular code C with intersection array* (3),

$$|C| = \bar{v}/(1 + \kappa_1 + \cdots + \kappa_t), \tag{5}$$

$$k_i = \kappa_i\,|C|, \tag{6}$$

where

$$\kappa_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i} \quad (i = 1, \ldots, t). \tag{7}$$

Proof. The number of codewords is $|C| = k_0$, and the number of edges between subconstituents C_l and C_{l+1} is $k_l b_l = k_{l+1}c_{l+1}$. Using (6) to define κ_i we find $\kappa_0 = 1$ and $\kappa_{i+1} = \kappa_i b_i / c_{i+1}$ which yields (7). Since the number of vertices is $\bar{v} = k_0 + \cdots + k_t$, (5) follows. □

This generalizes the so-called *sphere-packing condition* for perfect codes, cf. van Lint [5]; the fact that $|C|$ and the k_i must be integral restricts the possibilities for the intersection array. Somewhat stronger (since $k_i = \sum_l p^l_{ij}$) are the following integrality conditions which hold when H is distance regular.

Theorem 3.2. *Let C be a completely regular code in a distance-regular graph H. Then there are constants* p^l_{ij} such that

$$|H_i(x) \cap C_j| = p^l_{ij} \quad \text{for all } x \in C_l; \tag{8}$$

they are computable from $p^l_{0j} = \delta_{jl}$, $p^l_{1j} = L_{lj}$ *and the recurrence relation*

$$p^{l-1}_{ij} c_l + p^l_{ij} a_l + p^{l+1}_{ij} b_l = \bar{c}_{i+1} p^l_{i+1\,j} + \bar{a}_i p^l_{ij} + \bar{b}_{i-1} p^l_{i-1\,j}.$$

(Here a_l, b_l, c_l *refer to parameters of C, and* $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$ *to those of H.)*

Proof. Clearly, the characteristic vectors g_l defined in the proof of Theorem 2.1 are linearly independent. Let V be the vector space spanned by the g_l. Then (4) implies that A maps V into itself. We now use the simple fact [1, 2] that for a distance regular graph,

$$AA_i = \bar{c}_{i+1}A_{i+1} + \bar{a}_i A_i + \bar{b}_{i-1}A_{i-1}. \tag{9}$$

This implies that the A_i are polynomials in A, hence they also map V into itself. Thus

$$A_i g_j = \sum_l p^l_{ij} g_l \tag{10}$$

for suitable numbers p^l_{ij}. For $x \in C_l$, this implies

$$p^l_{ij} = (A_i g_j)_x = \sum_y (A_i)_{xy}(g_j)_y = |\Gamma_i(x) \cap C_j|.$$

This proves (8), and a comparison with (1) yields $p^l_{1j} = L_{lj}$. Clearly $P^l_{0j} = \delta_{jl}$. Finally, the recurrence relation follows from (4), (9) and (10) by comparing the coefficients of

$$A(A_i g_j) = A\Big(\sum_l p^l_{ij} g_l\Big) = \sum_l p^l_{ij}(c_{l+1}g_{l+1} + a_l g_l + b_{l-1}g_{l-1})$$

and

$$(AA_i)g_j = (\bar{c}_{i+1}A_{i+1} + \bar{a}_i A_i + \bar{b}_{i-1}A_{i-1})g_j$$
$$= \sum_l (\bar{c}_{i+1} p^l_{i+1\,j} g_l + \bar{a}_i p^l_{ij} g_l + \bar{b}_{i-1} p^l_{i-1\,j} g_l). \qquad □$$

(Again (8), but not the recurrence relation, generalizes to arbitrary equitable partitions of a distance-regular graph.)

4. Geometric properties

Often, it is easier to check complete regularity in terms of the numbers

$$\lambda_i := p_{i0}^{i-1} \quad \text{and} \quad \mu_i := p_{i0}^{i}.$$

Theorem 4.1. *Let H be a distance regular graph with intersection array $\{\bar{b}_0, \ldots, \bar{b}_{d-1}; \bar{c}_1, \ldots, \bar{c}_d\}$. A code C in H with covering radius t is completely regular iff the condition:*

(C_i) *If $d(x, C) = i$ then C contains exactly μ_i codewords at distance i from x and λ_{i+1} codewords at distance $i+1$ from x*

holds for all i. In this case, the intersection array of C is given by

$$b_i = \bar{b}_i + \bar{c}_i - \frac{\lambda_{i+1}\bar{c}_{i+1}}{\mu_i} + \frac{(\lambda_i + \mu_i)c_i}{\mu_{i-1}} \quad (i < t), \qquad c_i = \frac{\mu_i \bar{c}_i}{\mu_{i-1}} \quad (i \le t). \tag{11}$$

Proof. If C is completely regular then (C) holds with $\lambda_i = p_{i0}^{i-1}$ and $\mu_i = p_{i0}^{i}$. Conversely, suppose that (C) holds, and fix $x \in C_i$. We count in two ways (y first, z first) the number of pairs $(y, z) \in H(x) \times C$ satisfying $d(y, z) = l$.

For $l = i - 1$ we must have $y \in C_{i-1}$, $z \in H_i(x)$, so that we get $|H(x) \cap C_{i-1}|\, \mu_{i-1} = \mu_i \bar{c}_i$. Hence $|H(x) \cap C_{i-1}|$ is the constant c_i given by (11). And for $l = i$ we must have $y \in C_{i-1} \cup C_i$, $z \in H_i(x) \cup H_{i+1}(x)$, and we get

$$c_i\lambda_i + |H(x) \cap C_i|\, \mu_i = \mu_i \bar{a}_i + \lambda_{i+1}\bar{c}_{i+1}$$

whence

$$|H(x) \cap C_i| = (\mu_i \bar{a}_i + \lambda_{i+1}\bar{c}_{i+1} - c_i\lambda_i)/\mu_i =: a_i$$

is constant. This implies that

$$|H(x) \cap C_{i+1}| = \bar{k} - a_i - c_i = b_i \quad (\text{use } \bar{k} = \bar{a}_i + \bar{b}_i + \bar{c}_i)$$

is constant, and shows that C is completely regular. $\square$

For example, the *repetition code* $C = \{0^n, 1^n\}$ is a completely regular code in the n-cube, with covering radius $t = \lceil n/2 \rceil$ and $\lambda_i = 0$ $(i = 1, \ldots, t)$, $\mu_i = 2$ if $2i = n$ and $\mu_i = 1$ otherwise. This trivial example has analogues in some codes with large *minimum distance*

$$d_C := \min\{d(x, y) \mid x, y \in C, x \neq y\}.$$

Proposition 4.2. *Let C be a code in a graph H satisfying for all i the condition (C_i) of Theorem 4.1. Then*

$$d_C \le 2t_C + 1, \tag{12}$$

$$\lambda_i = 0 \quad \text{if } 2i \le d_C, \tag{13}$$

$$\mu_i = 1 \quad \text{if } 2i + 1 \le d_C, \tag{14}$$

$$\mu_i \ge \mu_{i-1}. \tag{15}$$

Proof. (i) (12) is clear if t_C equals the maximal distance in H. If $t := t_C$ is smaller, choose $x \in C$, and z at distance $t + 1$ from x. By definition of t, there is a vertex $y \in C$ at distance $\le t$ from z, and since $x \neq y$ we have $d_C \le d(x, y) \le d(x, z) + d(y, z) \le 2t + 1$. Thus (12) holds.

(ii) Let $d(x, C) = i$. Then there is a codeword $z \in C$ at distance i from x. If $\mu_i > 1$ then there is another such codeword y and $d_C \le d(z, y) \le d(z, x) + d(y, x) = 2i$, which gives (14). And if $\lambda_{i+1} > 0$ then there is a codeword $y \in C$ at distance $i + 1$ from x, so that $d_C \le d(z, y) \le d(z, x) + d(y, x) = 2i + 1$. With $i - 1$ in place of i, this yields (13).

(iii) Let x' be the neighbour of x on a geodesic from x to z, so that $d(x', z) = i - 1$. Then the μ_{i-1} codewords at distance $i - 1$ from x' have distance i from x, giving (15). $\square$

The extremal case in (12) is very special.

Theorem 4.3. *For a code C in a distance regular graph H, the following properties are equivalent:*

(P1) $d_C = 2t_c + 1$,

(P2) *C is completely regular with $\lambda_i = 0$, $\mu_i = 1$ $(i = 0, \ldots, t_C)$,*

(P3) *C is perfect, i.e., the spheres of some fixed radius e around codewords partition H. In particular, Lloyd's theorem holds for perfect codes.*

Proof. (P1)$\Leftrightarrow$(P3): Spheres of radius t_c cover H, and they are disjoint iff $d_C \ge 2t_C + 1$, hence by (12) iff (P1) holds.

(P1)$\Rightarrow$(P2). By (13) and (14), $\lambda_1 = 0$, $\mu_i = 1$ for $i = 0, \ldots, t_C$, and by Theorem 4.1, this implies that C is completely regular.

(P2)$\Rightarrow$(P1): Suppose that $d_C \le 2t_C$, let $x, y \in C$ have distance d_C, and let z be a vertex on a geodesic between x and y at distance i from x and $j = d - i$ from y. For the choice $i = [d/2]$ we have $j \in \{i, i+1\}$, and $d(z, C) = i$ since d_C is minimal. But $j \neq i$ since $\mu_i = 1$ and $j \neq i + 1$ since $\lambda_{i+1} = 0$, contradiction. Hence $d_C \ge 2t_C + 1$, and equality holds by (12). $\square$

5. Open problems

While the classification of perfect codes with large covering radius has been completed for the Hamming cubes $H(n, q)$—with major contributions of van Lint [5]—the corresponding problem for completely regular codes is wide open. The survey in [2] shows that a number of infinite families with unbounded covering radius exist, so a complete classification will be much more difficult. However, the known examples are compatible with the following.

Conjecture. The only completely regular code C in $H(n, q)$ with $|C| > 2$ and $d_C \geqslant 8$ is the extended binary Golay code ($n = 24$, $q = 2$, $d_C = 8$).

Apart from settling this conjecture, some further problems are of interest:

(1) Find more feasibility conditions for the λ_i, μ_i.

(2) Find all 'small' completely regular codes in the Hamming cubes ($qn \leqslant 48$ is a natural bound).

(3) Characterize the known competely regular codes [2] by their parameters.

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