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Krein Conditions and Near Polygons

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In this article we present a new proof for (and a generalization of) the Krein condition for association schemes. The proof yields necessary and sufficient conditions for the case of equality. In the special case of regular near polygons we give a second matrix-free proof of the special Krein condition $q_{dd}^d \geq 0$ and a corresponding characterization of the equality case. Also, Mathon's inequality for near hexagons is generalized to arbitrary regular near polygons. © 1990 Academic Press, Inc.

1. THE KREIN CONDITIONS

Let X be a finite set. An *association scheme with d classes* is a pair $(X, \mathscr{R})$ such that

(i) $\mathscr{R} = \{R_0, R_1, ..., R_d\}$ is a partition of $X \times X$;

(ii) $R_0 = \Delta := \{(x, x) \mid x \in X\}$;

(iii) $(x, y) \in R_i \Rightarrow (y, x) \in R_i$;

(iv) there are numbers p_{ij}^l (the intersection numbers of the scheme) such that for any pair $(x, y) \in R_l$ the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^l.

The number $k_i := p_{ii}^0$ of $z \in X$ with $(x, z) \in R_i$(which is independent of $x \in X$) is called the *valency* of R_i, and the total number of points of X is

$$v := |X| = \sum_{i=0}^{d} k_i.$$

The $n \times n$ matrices A_i (indexed by $X \times X$) whose (x, y)–entry is 1 if $(x, y) \in R_i$, and 0 otherwise, span a real algebra $\mathfrak{A}$ closed under both ordinary multiplication $(A, B) \to AB$ and Hadamard (elementwise) multiplication $(A, B) \to A \circ B$, called the *Bose–Mesner-algebra* of the scheme. The

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Bose–Mesner-algebra has a basis of minimal idempotents E_j, which we write as

$$E_j = \frac{1}{v} \sum_{i=0}^{d} Q_{ij} A_i \qquad (j = 0, ..., d). \tag{1}$$

The ranks $f_j = \mathrm{rk}(E_j) = \mathrm{tr}(E_j)$ are called the *multiplicities* of the scheme. It is a well-known fact that the Q_{ij} and f_j can be computed from the p^l_{ij} (Bose and Mesner [3]); the fact that all f_j must be integral severely restricts the possibilities for the p^l_{ij}. Further restrictions on the parameters, known as the *Krein conditions*, are due to Scott [10]. We derive them here in a new way, and characterize the equality case. Let $\Sigma(A)$ denote the sum of all entries of the matrix A, and define q^k_{ij} by

$$E_i \circ E_j = \frac{1}{v} \sum_k q^k_{ij} E_k.$$

1.1. Theorem. *For all* $i, j, k \in \{0, ..., d\}$.

$$v f_k q^k_{ij} = \sum_{l=0}^{d} k_l Q_{li} Q_{lj} Q_{lk} \geqslant 0, \tag{2}$$

with equality if and only if

$$\sum_{x \in X} E_i(u, x) E_j(v, x) E_k(w, x) = 0 \qquad \textit{for all} \quad u, v, w \in X. \tag{3}$$

Proof. Since the E_i are symmetric idempotent matrices,

$$\sum_{u \in X} E_i(u, x) E_i(u, y) = E_i(x, y). \tag{4}$$

Hence, if we denote the left hand side of (3) by $q(u, v, w)$ we have

$$\begin{aligned} \Sigma(E_i \circ E_j \circ E_k) &= \sum_{x, y \in X} E_i(x, y) E_j(x, y) E_k(x, y) \\ &= \sum_{u, v, w \in X} \left(\sum_{x \in E} E_i(u, x) E_j(v, x) E_k(w, x) \right) \\ &\quad \times \left(\sum_{y \in X} E_i(u, y) E_j(v, y) E_k(w, y) \right) \\ &= \sum_{u, v, w \in X} q(u, v, w)^2 \geqslant 0 \end{aligned}$$

by (4). Since both

$$\begin{aligned} v^2 \Sigma(E_i \circ E_j \circ E_k) &= v^2 \cdot \mathrm{tr}((E_i \circ E_j) E_k) = v \cdot \mathrm{tr}\left(\sum_l q^l_{ij} E_l E_k \right) \\ &= v \cdot \mathrm{tr}(q^k_{ij} E_k) = v f_k q^k_{ij} \end{aligned}$$

and (by (1))

$$v^2 \Sigma(E_i \circ E_j \circ E_k) = \frac{1}{v} \sum_l Q_{li} Q_{lj} Q_{lk} \Sigma(A_l) = \sum_l k_l Q_{li} Q_{lj} Q_{lk},$$

inequality (2) holds, and equality holds in (2) if and only if $\sum q(\alpha, \beta, \gamma)^2 = 0$, which is equivalent to (3). ∎

1.2. *Remark.* The argument of the proof can be applied to

$$\sum_x E_{i_1}(u_1, x) \cdots E_{i_s}(u_s, x)$$

and

$$\begin{aligned} &\sum_{x_1, x_2} E_{i_1}(u_1, x_1) \cdots E_{i_s}(u_s, x_1) \\ &\quad \times (E_{j_1}(v_1 x_1) \cdots E_{j_t}(v_t, x_1) - E_{j_1}(v_1, x_2) \cdots E_{j_t}(v_t, x_2)) \end{aligned}$$

in place of $q(u, v, w)$. This results in the inequalities

$$\sigma_{i_1 \cdots i_s} := \sum_{l=0}^{d} k_k Q_{li_1} \cdots Q_{li_s} = v^{s-1} \Sigma(E_{i_1} \circ \cdots \circ E_{i_s}) \geqslant 0 \tag{2a}$$

and

$$\sigma_{i_1 \cdots i_s j_1 \cdots j_t} \geqslant \frac{1}{v} \sigma_{i_1 \cdots i_s} \sigma_{j_1 \cdots j_t}. \tag{2b}$$

Biggs [2] mentions a remark by Cameron and Delsarte that (2a) follows from (2) and hence imposes no new restrictions on the parameters. It is not known whether (2b) excludes any otherwise feasible parameter sets.

2. A Special Case

Bill Kantor asked me to write down a matrix–free proof of the nontrivial Krein condition

$$t \leqslant s^2 \qquad \text{if} \quad s > 1$$

for generalized octagons, together with a characterization of the equality case. This will be done in the following in the more general context of regular near polygons. A partial linear space $(X, \mathscr{L})$ is called a *near polygon* if for every point $x \in X$ and every line $L \in \mathscr{L}$ there is a unique point on L nearest to x. A *regular near polygon* (*2d-gon*) is a near polygon whose point graph is distance-regular (of diameter d), i.e., if for any two points $x, y \in X$ at distance i the number of neighbours of y at distance j from x depends only on j, and hence is a constant c_i, a_i, or b_i for $j = i-1, i, i+1$, respectively. In particular, this implies that every line has constant size $s+1$, every point is in a constant number $t+1$ of lines, and the relations

$$a_i = (s-1)\,c_i, \qquad b_i = s(t+1-c_i) \qquad (i = 0, ..., d) \tag{5}$$

hold. Moreover,

$$c_0 = 0, \qquad c_1 = 1, \qquad c_d = t+1. \tag{6}$$

For these and other properties of near polygons, see Shult and Yanushka [11] and Brouwer and Wilbrink [5]. If (1) holds with $c_i = 1$ for $i < d$ then $(X, \mathscr{L})$ is called a *generalized* 2*d-gon*.

We shall need the following auxiliary results.

2.1. Lemma. *Let $(X, \mathscr{L})$ be a regular near polygon.*

(i) *The number of points at distance i from $x \in X$ is a constant k_i independent of x, and we have $k_0 = 1$,*

$$k_i c_i = k_{i-1}\, b_{i-1} \qquad (i = 1, ..., d). \tag{7}$$

(ii) *For $x, y \in X$ at distance l, the number of $z \in X$ with $d(z, x) = i$, $d(z, y) = j$ is a constant p^l_{ij} independent of x and y.*

(iii) *We have*

$$\sum_{i,j=0}^{d} \left(-\frac{1}{s}\right)^{i+j} p^l_{ij} = \left(-\frac{1}{s}\right)^{l} \sigma \qquad (l = 0, ..., d),$$

where $\sigma = \sum_{j=0}^{d} s^{-2j} k_j$.

Proof. (i) By counting in two ways the number of edges yz with $d(x, y) = i$, $d(x, z) = i-1$ we find (7), and thus inductively that k_i is constant.

(ii) This holds for $j = 0$ with $p^l_{i0} = \delta_{il}$. Assuming the validity for all $j' \leq j$ in place of j we fix $x, y \in X$ at distance l and count in two ways the number of pairs $(v, w) \in X \times X$ such that $d(v, w) = j$, $d(v, x) = i$, and $d(w, y) = 1$. We find

$$c_l p^{l-1}_{ij} + a_l p^l_{ij} + b_l p^{l+1}_{ij} = c_{j+1} p^l_{ij+1} + a_j p^l_{ij} + b_{j-1} p^l_{ij-1}. \tag{8}$$

Since $c_{j+1} \neq 0$ this shows that p^l_{ij+1} is also independent of x and y. Thus (ii) follows by induction.

(iii) Define $\pi^l_j := \sum_i (-1/s)^i p^l_{ij}$. Then multiplication of (4) by $(-1/s)^i$ and summation over i gives

$$c_l \pi^{l-1}_j + a_j \pi^l_j + b_l \pi^{l+1}_j = c_{j+1} \pi^l_{j+1} + a_j \pi^l_j + b_{i-1} \pi^l_{j-1}.$$

Now a short calculation using (1) and (3) establishes inductively that

$$\pi^l_j = \left(-\frac{1}{s}\right)^{j+l} k_j$$

whence

$$\sum_{i,j} \left(-\frac{1}{s}\right)^{i+j} p^l_{ij} = \sum_j \left(-\frac{1}{s}\right)^{j} \pi^l_j = \left(-\frac{1}{s}\right)^{l} \sum_j \left(-\frac{1}{s}\right)^{2j} k_j = \left(-\frac{1}{s}\right)^{l} \sigma. \quad \blacksquare$$

2.2. *Remark.* Parts (i) and (ii) hold for arbitrary distance regular graphs but are usually proved by matrix arguments; see, e.g., Biggs [1]. Part (iii) shows that for regular near polygons the matrix

$$E = \sigma^{-1} \sum_{i=0}^{d} \left(-\frac{1}{s}\right)^{i} A_i$$

is idempotent; indeed, the equations $A_i A_j = \sum p^l_{ij} A_l$ together with (iii) imply that $E^2 = E$. Hence the only eigenvalues of E are 0 and 1, and $\mathrm{rk}(E) = \mathrm{tr}(E) = \sigma^{-1} v$. Since one easily checks that $A_1 E = (-1-t) E$ we find that $\theta_d = -1-t$ is an eigenvalue of A_1 of multiplicity $f_d = \sigma^{-1} v$, and E is the corresponding idempotent.

Now let $a, b, c \in X$. We denote by $q_l(a, b, c)$ the number of points $x \in X$ such that $d(a, x) + d(b, x) + d(c, x) = l$ and write

$$q(a, b, c) := \sum_l \left(-\frac{1}{s}\right)^{l} q_l(a, b, c). \tag{9}$$

2.3. Theorem. *For a regular near polygon with line size $s+1$ we have*

$$\chi := \sum_{l=0}^{d} \left(-\frac{1}{s}\right)^{3l} k_l \geq 0 \tag{10}$$

with equality iff $q(a, b, c) = 0$ for all $a, b, c \in X$.

Proof. The sum

$$\sum_{a,b,c} q_i(a,b,c)\, q_j(a,b,c)$$

equals the number of $(x, y, a, b, c) \in X^s$ *such that* $d(a,x)+d(b,x)+d(c,x)=i$ and $d(a,y)+d(b,y)+d(c,y)=j$; therefore it equals the sum

$$\sum_l vk_l \Big(\sum_{\substack{i_1+i_2+i_3=i\\ j_1+j_2+j_3=j}} p^l_{i_1j_1} p^l_{i_2j_2} p^l_{i_3j_3} \Big).$$

Therefore

$$\begin{aligned} 0 \leqslant \sum_{a,b,c} q(a,b,c)^2 &= \sum_{a,b,c} \sum_{i,j} \left(-\frac{1}{s}\right)^{i+j} q_i(a,b,c)\, q_j(a,b,c) \\ &= \sum_l vk_l \sum_{\substack{i_1,i_2,i_3\\ j_1,j_2,j_3}} \left(-\frac{1}{s}\right)^{i_1+i_2+i_3+j_1+j_2+j_3} p^l_{i_1j_1} p^l_{i_2j_2} p^l_{i_3j_3} \\ &= \sum_l vk_l \Big(\sum_{i,j} \left(-\frac{1}{s}\right)^{i+j} p^l_{ij} \Big)^3 = v\sigma^3 \sum_l \left(-\frac{1}{s}\right)^{3l} = v\sigma^3\chi. \end{aligned}$$

Since $\sigma > 0$, this imples $\chi \geqslant 0$. Moreover, $\chi = 0$ implies $\sum q(a,b,c)^2 = 0$ and hence $q(a,b,c)=0$ for all a, b, c. ∎

2.4. *Remark.* The inequality (10), and its consequence (13) below, were derived from the Krein condition by Brouwer and Wilbrink [5]. Indeed, comparing Remark 2.2 with (1) we find that $Q_{id} = (-1/s)^i f_d$ for regular near polygons. Hence we have $vf_d q^d_{dd} = \sum k_i Q^3_{id} = f^3_d \chi$, so that the above inequality is the Krein condition $q^d_{dd} \geqslant 0$. The expression (9) is just the left hand side of (3) for $i=j=h=d$. It is not difficult to provide in a similar way counting proofs for the other Krein conditions provided the corresponding Q_{ij} are known, but since the other Q_{ij} are not natural in the matrix-free setting we give no details.

In the small diameter case it is now an easy matter to compute χ from (5), (7), and (10).

For generalized quadrangles $(d=2)$ we find $\chi = (s^2-1)(s^2-t)/s^4$, whence

$$t \leqslant s^2 \qquad \text{if} \quad s > 1. \tag{11}$$

Examples with equality are the generalized quadrangles from $U_5(q)$ with $s=q$, $t=q^2$. (See, e.g., Payne and Thas [9].)

For generalized quadrangles with $t=s^2$, the condition $q(a,b,c)=0$ is trivial unless a, b, c have mutual distance 2; in this case it is easily found to be equivalent to the condition that the number of points adjacent with a, b, and c is constant (Cameron, Goethals, and Seidel [6]). For generalized octagons $(d=4,\ c_2=c_3=1)$ we find

$$\chi = (s^2-1)(s^2-t)(s^4+t^2)/s^8,$$

whence again (11) holds. Examples with equality are the generalized octagons from ${}^2F_4(2^{2k+1})$ with $s=2^{2k+1}$, $t=2^{4k+2}$ (see, e.g., [4]). Again, for $t=s^2$, the conditions $q(a,b,c)=0$ seems to be nontrivial only when all distances between a, b, c are $\geqslant 2$; but it is not clear whether they simplify to natural geometric properties.

For generalized hexagons $(d=3,\ c_2=1)$ we find no nontrivial restriction. However, the Krein condition $q^2_{33} \geqslant 0$ yields $t \leqslant s^3$ if $s>1$ (Haemers and Roos [7]). Examples with equality are the generalized hexagons from ${}^3D_4(q)$ with $s=q$, $t=q^3$ (see, e.g., [4]).

More generally, $q^2_{22} \geqslant 0$ gives for near hexagons $(d=3)$ the inequality

$$t \leqslant s^3 + (c_2-1)(s^2-s+1) \qquad \text{if} \quad s>1 \tag{12}$$

of Mathon [8] (see Brouwer and Wilbrink [5] and Sect. 3 below). The inequality $\chi \geqslant 0$ (i.e., $q^3_{33} \geqslant 0$) gives for near hexagons the restriction

$$c_2 \leqslant \frac{t(t+1)}{t+s^2(t-s^2)} \qquad \text{if} \quad t \geqslant s^2 > 1. \tag{13}$$

(12) and (13) together imply the inequality

$$t \leqslant s^2(s^2+1) \qquad \text{if} \quad s>1. \tag{14}$$

Equality in (14) implies that (12) and (13) hold with equality, and $c_2 = s^2+1$. By Shult and Yanushka [11], near hexagons with $c_2=s^2+1$, $c_3=t+1=s^4+s^2+1$ are dual polar spaces from $U_7(q)$, with $q=s$ a prime power. In particular, all near hexagons with equality in (14) are already known.

3. Generalized Mathon Inequalities for Regular Near Polygons

In this section we give a proof of a generalization of Mathon's inequality (12) which avoids eigenvalue calculations. However, for lack of a geometric argument we use a small amount of lincar algebra.

For a set S of points (or a single point $S=x$) we write $\bar{S}$ for the sum of the columns indexed by S of the matrix

$$B=\sigma^{-1/2}\sum_{i=0}^{d}\left(-\frac{1}{s}\right)^{i}A_i.$$

By Remark 2.2, $B=\sigma^{1/2}E$ satisfies $B^TB=B^2=\sigma E^2=\sigma E$; hence the standard inner product of two columns of B is

$$(\bar{x},\bar{y})=u_i:=\left(-\frac{1}{s}\right)^{i} \quad \text{when} \quad d(x,y)=i. \tag{15}$$

We shall use this relation to prove

3.1. THEOREM. *The parameters of a regular near polygon with $s>1$ satisfy*

$$c_i\leqslant(s^i+1)(c_{i-1}+s^{i-2})/(s^{i-2}+1) \qquad \textit{for odd} \quad i>2, \tag{16a}$$

$$c_i\geqslant(s^i-1)(c_{i-1}-s^{i-2})/(s^{i-2}-2) \qquad \textit{for even} \quad i>2. \tag{16b}$$

Proof. Let x_1, x_2 be points at distance i, and let $C_l(l=1,2)$ be the set of points adjacent with x_l and at distance $i-1$ from x_{3-l}. Clearly C_1 and C_2 are cocliques of size c_i. A routine calculation using (15) gives

$$(\bar{x}-\bar{y},\bar{x}-\bar{y})=2(u_0-u_i),$$
$$(\bar{x},\bar{C}-\bar{D})=(-\bar{y},\bar{C}-\bar{D})=c_i(u_1-u_{i-1}),$$
$$(\bar{C},\bar{C})=(\bar{D},\bar{D})=c_i(u_0+(c_i-1)\,u_2),$$
$$(\bar{C},\bar{D})=c_i(c_{i-1}u_{i-2}+(c_i-c_{i-1})\,u_i).$$

Hence the Cauchy–Schwarz inequality

$$(\bar{x}-\bar{y},\bar{C}-\bar{D})^2\leqslant(\bar{x}-\bar{y},\bar{x}-\bar{y})(\bar{C}-\bar{D},\bar{C}-\bar{D})$$

gives

$$\{2c_i(u_i-u_{i-1})\}^2\leqslant 2(u_0-u_i)$$
$$\cdot 2c_i\{u_0+(c_i-1)\,u_2-c_{i-1}u_{i-2}-(c_i-c_{i-1})\,u_i\}$$

which simplifies to

$$c_i\{(u_1-u_{i-1})^2-(u_0-u_i)(u_2-u_i)\}\leqslant(u_0-u_i)\{c_{i-1}(u_i-u_{i-2})+u_0-u_2\}.$$

Inserting the values (15) for the u_i gives, after division by $s^{-2i}(s^2-1)>0$, the inequality

$$(1-(-s)^{i-2})\,c_i\leqslant(1-(-s)^i)(c_{i-1}-(-s)^{i-2}).$$

This implies the assertion. ∎

3.2. *Remarks.* (i) For $i=3$, (16a) reduces to the Mathon inequality (12).

(ii) The above generalized Mathon inequalities are the simplest case of a more general set of inequalities for arbitrary distance regular graph given by Terwilliger [12]. In particular, his results imply that equality for all i in Theorem 3.1 is equivalent to the so-called Q-polynomial property with respect to the idempotent E.

The only known examples of this situation (cf. [4]) are the unitary dual polar spaces, the generalized hexagons associated with ${}^3D_4(q)$, and two sporadic near polygons with 729 and 759 points defined in terms of the Golay codes (see [11]).

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