

# The Graphs with Spectral Radius Between 2 and $\sqrt{2 + \sqrt{5}}$

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

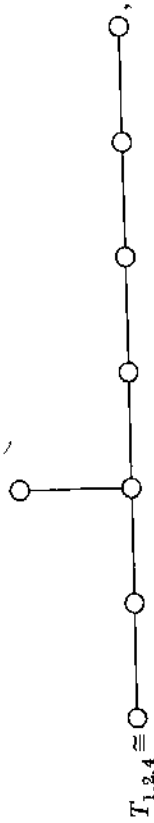
Submitted by Alexander Schrijver

## ABSTRACT

We complete the determination of the graphs in the title, begun by Cvetković, Doob, and Gutman.

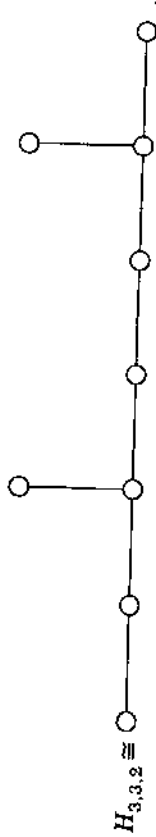
The spectral radius of a graph is the largest eigenvalue of its  $(0, 1)$  adjacency matrix. Hoffman [3] shows that graphs  $G$  properly containing a circuit have largest eigenvalue  $\lambda_{\max}(G) > \tau^{3/2} = \sqrt{2 + \sqrt{5}} \approx 2.058171$  [where  $\tau = (1 + \sqrt{5})/2$ ] and that  $\tau^{3/2}$  is a limit point of these numbers  $\lambda_{\max}(G)$ . Cvetković, Doob, and Gutman [1] classify the graphs  $G$  with  $2 < \lambda_{\max}(G) \leq \tau^{3/2}$  and find that these are certain trees without vertices of degree at least 4, and with at most two vertices of degree 3.

More explicitly, let  $T_{i,j,k}$  be the graph with  $i + j + k + 1$  vertices consisting of three paths with  $i$ ,  $j$ , and  $k$  edges, respectively, where these paths have one end vertex in common, e.g.,



$T_{1,2,4} \cong$

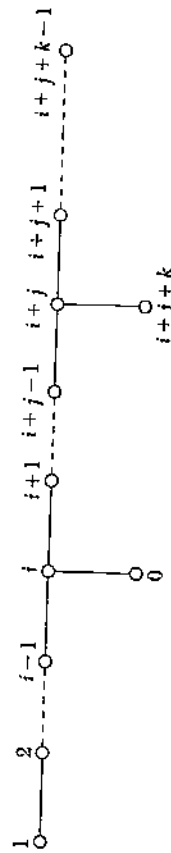
$T_{1,2,4}$  is the graph more commonly known as  $E_8$ . Also, let  $H_{i,j,k}$  be the graph with  $i+j+k+1$  vertices consisting of a path with  $u=i+j+k-1$  vertices  $x_1, \dots, x_u$  with two extra edges affixed at  $x_i$  and  $x_{u+1-k}$ , e.g.,



The abovementioned authors show that if  $2 < \lambda_{\max}(G) \leq \tau^{3/2}$ , then  $G$  is one of the graphs  $T_{i,j,k}$  or  $H_{i,j,k}$ ; furthermore, that the  $T_{i,j,k}$  occurring are  $T_{1,2,k}$  ( $k \geq 6$ ),  $T_{1,3,k}$  ( $k \geq 4$ ),  $T_{1,j,k}$  ( $k \geq j \geq 4$ ),  $T_{2,2,k}$  ( $k \geq 3$ ), and  $T_{2,3,3}$ , and that each of these has largest eigenvalue less than  $\tau^{3/2}$ . Concerning the  $H_{i,j,k}$ , they did not succeed in determining the precise triples  $(i, j, k)$  occurring, but gave computer results for small  $(i, j, k)$ . However, it is not difficult to determine for which  $i, j, k$  one has  $\lambda_{\max}(H_{i,j,k}) \leq \tau^{3/2}$ . On the one hand, this can easily be read off from the explicit formulae for the characteristic polynomial of  $H_{i,j,k}$  given in Goodman, de la Harpe, and Jones [2]; on the other hand, it follows immediately from the results in Neumaier [4]. Here, we shall follow the latter approach.

**PROPOSITION.**  $\lambda_{\max}(H_{i,j,k}) \leq \tau^{3/2}$  if and only if  $\tau^j \geq (\tau^i - 2)(\tau^k - 2)$ , and equality does not occur.

*Proof.* We apply Theorem 2.4 of [4]. We have to construct a partial  $\lambda$ -eigenvector for  $\lambda = \tau^{3/2} = \mu + \mu^{-1}$ , where  $\mu = \tau^{1/2}$ . Label the vertices of  $H_{i,j,k}$  as follows:



and define a vector  $e$  with components

$$e_l = \frac{\mu^l - \mu^{-l}}{\alpha} \quad (1 \leq l \leq i),$$

$$e_0 = e_i(\mu - \mu^{-1}),$$

$$e_{i+l} = \frac{\mu^{i-l} + \mu^{l-i} - 2\mu^{-i-l}}{\alpha} \quad (0 \leq l \leq j),$$

$$e_{i+j+k} = \mu - \mu^{-1},$$

$$e_{i+j+l} = \frac{\mu^{k-l} - \mu^{l-k}}{\beta} \quad (0 \leq l < k),$$

where

$$\alpha = \mu^{i-j} + \mu^{j-i} - 2\mu^{-i-j}, \quad \beta = \mu^k - \mu^{-k}.$$

Using the relation  $\mu^2 - \mu^{-2} = 1$  (which holds because  $\tau^2 = \tau + 1$ ), one easily checks that  $e$  is a positive partial  $\lambda$ -eigenvector with respect to the vertex  $i+j$ , and the exit value is

$$\begin{aligned} \epsilon &= \lambda - e_{i+j+k} - e_{i+j-1} - e_{i+j+1} \\ &= 2\mu^{-1} - \frac{\mu^{i-j+1} + \mu^{j-i-1} - 2\mu^{-i-j+1}}{\mu^{i-j} + \mu^{j-i} - 2\mu^{-i-j}} - \frac{\mu^{k-1} - \mu^{1-k}}{\mu^k - \mu^{-k}}. \end{aligned}$$

Distributing one of the  $\mu^{-1}$  to each fraction gives

$$\epsilon = \frac{(\mu - \mu^{-1})(2 - \mu^{2i})}{\mu^{2i} + \mu^{2j} - 2} + \frac{\mu - \mu^{-1}}{\mu^{2k} - 1},$$

and since  $\mu^2 = \tau$ , this simplifies to

$$\epsilon = (\mu - \mu^{-1}) \frac{\tau^j - (\tau^i - 2)(\tau^k - 2)}{(\tau^i + \tau^j - 2)(\tau^k - 1)}.$$

By Theorem 2.4 of [4],  $\lambda_{\max}(H_{i,j,k}) \leq \lambda$  if and only if  $\epsilon \geq 0$ , and  $\lambda_{\max}(H_{i,j,k}) = \lambda$  if and only if  $\epsilon = 0$ . But one easily checks that the latter cannot happen. ■

Combining this with the results of Cvetković, Doob, and Gutman yields:

**THEOREM.** *Let  $G$  be a graph with  $2 < \lambda_{\max}(G) \leq \tau^{3/2}$ . Then  $G$  is one of the graphs  $T_{i,j,k}$  (see above), or one of the graphs  $H_{i,j,k}$ , where  $j \geq i + k$ , or  $i = 3$  and  $j \geq k + 2$ , or  $i = 2$  and  $j \geq k - 1$ , or  $(i, j, k)$  is one of  $(2, 1, 3)$ ,  $(3, 4, 3)$ ,  $(3, 5, 4)$ ,  $(4, 7, 4)$ ,  $(4, 8, 5)$ . None of these graphs has  $\lambda_{\max}(G) = \tau^{3/2}$ .*

Now we can answer a question posed in [2]:

**COROLLARY.** *The set of spectral radii of graphs is not a closed subset of the real line.*

**POSTSCRIPT**

This result was obtained by both authors independently (fall 1986) after having received a copy of [2] from J. J. Seidel. Shortly afterwards we learned from him that Shearer [5] had shown that each real  $\lambda \geq \tau^{3/2}$  is a limit point of spectral radii (from which our Corollary follows immediately, since spectral radii are algebraic integers) and that Godsil had remarked that  $\sqrt{2 + \sqrt{5}}$  cannot be a spectral radius, since otherwise all its conjugates would be eigenvalues, too, but  $\sqrt{2 - \sqrt{5}}$  is not real.

**REFERENCES**

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Received March 1988; final manuscript accepted 15 July 1988