Graph Representations, Two-Distance Sets, and Equiangular Lines

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

Using the new concepts of graph representations and heights, it is shown (among other results) that all sufficiently large sets of equiangular lines with mutual angle $\arccos\frac{1}{5}$ are known.

INTRODUCTION

Motivated by the classification of two-distance sets in Euclidean space, the concept of a spherical representation of a graph Γ is introduced. A graph Γ with adjacency matrix A has an $(r+t, r-1, r)$-representation iff $tI - A + rJ$ is positive semidefinite. Such representations are related to the eigenvalues of Γ and to another invariant, the t-height of Γ. A graph Γ with adjacency matrix A has t-height $h_t(\Gamma) = j^T x$ if $(tI - A)x = j$ has a solution, and $h_t(\Gamma) = \infty$ otherwise.

In Section 1 several results are proved which show how heights give insight into the structure of (induced) subgraphs of graphs with a representation. In particular, the structure of pillars is explored. The pillars of a subgraph Δ of Γ are the sets $P_a(\Delta)$ such that

$$\gamma \in P_a(\Delta) \quad \text{iff} \quad \Gamma(\gamma) \cap \Delta = \Delta_0;$$

here Δ_0 is a subgraph of Δ, and a is the characteristic vector of Δ_0. In Section 2, special representations corresponding to equiangular lines are considered. The main result is the following characterization of graphs with a

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large number of vertices which are switching equivalent to graphs with largest eigenvalue $\leqslant t$.

THEOREM. *Let $G_t^{\#}$ denote the class of finite graphs with largest eigenvalue $>t$ such that each proper induced subgraph has largest eigenvalue $\leqslant t$. If $G_t^{\#}$ is finite, then there is a number $v(t)$ such that every graph with more than $v(t)$ vertices having a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ representation is switching equivalent to a graph with largest eigenvalue $\leqslant t$.*

In particular, since $G_2^{\#}$ is finite (Koszul [6]) and all graphs with smallest eigenvalue $\leqslant 2$ are known (Smith [15]; precisely the reduced and extended Dynkin diagrams with simple bonds occur), this implies that all sufficiently large sets of equiangular lines with mutual angle $\arccos\frac{1}{5}$ are known. This complements investigations of Lemmens and Seidel [8].

In the following, graphs are finite, undirected, without loops or multiple edges. The adjacency relation is written as $\sim$. All subgraphs are understood to be induced subgraphs. The adjacency matrix of a graph Γ is the matrix $A = A_\Gamma$ indexed by the vertices of Γ with $A_{\gamma\delta}=1$ if $\gamma\sim\delta$, $A_{\gamma\delta}=0$ otherwise. The eigenvalues of A_Γ are called the eigenvalues of Γ; they are denoted in decreasing order by $\theta_1(\Gamma)\geqslant\theta_2(\Gamma)\geqslant\cdots\geqslant\theta_v(\Gamma)$, where v is the number of vertices of Γ and multiple eigenvalues are counted according to their multiplicities. In particular, $\theta_{\max}(\Gamma)$ is the largest and $\theta_{\min}(\Gamma)$ the smallest eigenvalue of Γ. The letters j (and J) denote a vector (and matrix) all of whose entries are equal to one.

1. SPHERICAL REPRESENTATIONS

Let Γ be a graph. A mapping $\bar{\ }:\Gamma\to\mathbb{R}^N$ is called a (spherical) (p,q,r)-*representation* of Γ if the inner product of the images $\bar\gamma,\bar\delta$ of any two vertices $\bar\gamma,\bar\delta\in\Gamma$ satisfies

$$(\bar\gamma,\bar\delta)=\begin{cases} p & \text{if } \gamma=\delta,\\ q & \text{if } \gamma\sim\delta,\\ r & \text{otherwise.}\end{cases}$$

Here $(x,y)=x_1y_1+\cdots+x_Ny_N$ denotes the standard inner product of $\mathbb{R}^N$.

The image of a graph under a (p,q,r)-representation is a spherical 2-distance set in $\mathbb{R}^N$ (cf. Larman, Rogers, and Seidel [7], Neumaier [10], Blokhuis [1]), and conversely, every spherical 2-distance set in $\mathbb{R}^N$ is obtained in this way. By scaling the images of a (p,q,r)-representation of Γ by

multiplication with $|r-q|^{-1/2}$, we obtain for $q<r$ an $(r'+t, r'-1, r')$-representation of Γ with

$$t=\frac{p-r}{r-q},\qquad r'=\frac{r}{r-q},$$

and for $q>r$ an $(r'+t,r'-1,r')$-representation of the complement of Γ with

$$t=\frac{p-q}{q-r},\qquad r'=\frac{q}{q-r}.$$

Hence, since the case $q=r$ is uninteresting, it is no restriction of generality to take the parameters of a spherical representation of the form $(p,q,r)=(r+t,r-1,r)$.

In the following we fix a value of $t>0$ and ask for possible $(r+t,r-1,r)$-representations. The Gram matrix (of inner products) of the image of a subset Δ of a graph with a fixed representation is denoted by G_Δ, and the adjacency matrix of the subgraph induced on Δ is denoted by A_Δ. Clearly we have

$$G_\Delta = tI - A_\Delta + rJ \tag{1.1}$$

in an $(r+t,r-1,r)$-representation; in particular, $tI-A_\Delta+rJ$ is positive semidefinite for all $\Delta\subseteq\Gamma$. Conversely, if $G=tI-A_\Gamma+rJ$ is positive semidefinite of rank N, then G is the Gram matrix of a set X of vectors of $\mathbb{R}^N$ which are in a natural one-to-one correspondence with the vertices of Γ; thus there is a mapping $\bar{\ }:\Gamma\to X\subseteq\mathbb{R}^N$ affording an $(r+t,r-1,r)$-representation of Γ. This correspondence allows us to decide whether a graph has an $(r+t,r-1,r)$-representation with given r,t. In particular, we get for $r=0$ and $r=1$:

PROPOSITION 1.1. *Let Γ be a graph, and let $\bar\Gamma$ be the complement of Γ.*

(i) *Γ has a $(t,-1,0)$-representation iff $\theta_{\max}(\Gamma)\leqslant t$.*
(ii) *Γ has a $(t+1,0,1)$-representation iff $\theta_{\min}(\bar\Gamma)\geqslant -1-t$.*

Proof. $\theta_{\max}(\Gamma)\leqslant t$ iff $tI-A_\Gamma$ is positive semidefinite, and $\theta_{\min}(\bar\Gamma)\geqslant -1-t$ iff $A_{\bar\Gamma}+(1+t)I=tI-A_\Gamma+J$ is positive semidefinite. ∎

For other values of r it is convenient to have a criterion based on the eigenvalues of Γ and another invariant, the height. The *t-height* of Γ is the number $h_t(\Gamma)$ defined by

$$h_t(\Gamma)=\begin{cases} j^Tx & \text{if } (tI-A_\Gamma)x=j \text{ has a solution,}\\ +\infty & \text{otherwise.}\end{cases}$$

Note that, since A_Γ is symmetric, $(tI-A_T)x=j$ has a solution iff every vector z with $A_\Gamma z=tz$ is orthogonal to j; thus j^Tx does not depend on the choice of x. In particular,

$$h_t(\Gamma)=j^T(tI-A_T)^{-1}j$$

if t is not an eigenvalue of Γ. A basic property of the height is given by

PROPOSITION 1.2. *Let Γ be a graph with an $(r+t,r-1,r)$-representation, and let Δ be a subgraph of Γ with $h_t(\Delta)\neq\infty$. Then $h_t(\Delta)\neq 0$ and*

$$r\geqslant-\frac{1}{h_t(\Delta)}. \tag{1.2}$$

Proof. Let x be a solution of $(tI-A_\Delta)x=j$. Then $G_\Delta x=j+rJx=[1+rh_t(\Delta)]j$ and $0\leqslant x^TG_\Delta x=h_t(\Delta)+rh_t(\Delta)^2$. If $h_t(\Delta)\neq 0$, this implies (1.2); if $h_t(\Delta)=0$, then $x^TG_\Delta x=0$ but $0\neq G_\Delta x=j$, contradicting the fact that G_Δ is positive semidefinite. ■

The height can be expressed in terms of the eigenvectors of A_Γ.

PROPOSITION 1.3. *Let $j=\sum_{i=1}^{v}a_ix^{(i)}$ be the expansion of j in an orthonormal basis $x^{(1)},\dots,x^{(v)}$ of eigenvectors of A_Γ, i.e., $A_\Gamma x^{(i)}=\theta_i(\Gamma)x^{(i)}$. Then*

$$h_t(\Gamma)=\sum_{a_i\neq 0}\frac{a_i^2}{t-\theta_i(\Gamma)}. \tag{1.3}$$

In particular, if $h_t(\Gamma)<0$ then $t<\theta_1(\Gamma)$.

Proof. We have $a_i=x^{(i)T}j$, hence $h_t(\Gamma)\neq\infty$ iff $a_i=0$ whenever $\theta_i(\Gamma)=t$. In this case the vector

$$x=\sum_{a_i\neq 0}\frac{a_i}{t-\theta_i(\Gamma)}x^{(i)} \tag{1.4}$$

satisfies $(tI-A_\Gamma)x=j$, and multiplication by j^T gives (1.3). ■

THEOREM 1.4. *A graph Γ has an $(r+t,r-1,r)$-representation iff one of the following holds:*

(i) $r<0$, $\theta_1(\Gamma)<t$, $h_t(\Gamma)\leqslant-1/r$.
(ii) $r\geqslant 0$, $\theta_1(\Gamma)\leqslant t$.
(iii) $r>0$, $\theta_2(\Gamma)\leqslant t<\theta_1(\Gamma)$, $h_t(\Gamma)\leqslant-1/r$.

Proof. Γ has an $(r+t,r-1,r)$-representation iff $tI-A_\Gamma+rJ$ is positive semidefinite, i.e. iff $z^T(tI-A_\Gamma+rJ)z\geqslant 0$ for all $z\neq 0$. Expanding z in an orthonormal eigenvector basis (notation as in Proposition 1.3) $z=\sum b_ix^{(i)}$, we see that this is equivalent with

$$\sum b_i^2[t-\theta_i(\Gamma)]+r\left(\sum a_ib_i\right)^2\geqslant 0 \qquad \text{for all} \quad b_1,\dots,b_v\in\mathbb{R},$$

and this is equivalent with the requirement that all principal minors of the matrix $tI-\Omega+raa^T$ [where Ω is the diagonal matrix with diagonal entries $\Omega_{ii}=\theta_i:=\theta_i(\Gamma)$ and $a=(a_1,\dots,a_v)^T$] are nonnegative.

By calculating the values of these minors one finds that this is equivalent with

$$\prod_{i\in S}(t-\theta_i)+r\sum_{i\in S}a_i^2\prod_{\substack{j\neq i\\ j\in S}}(t-\theta_j)\geqslant 0 \qquad \text{for all} \quad S\subseteq\{1,\dots,v\}. \tag{1.5}$$

In particular, (1.5) implies for $S=\{i\}$ and $S=\{i,k\}$ $(i\neq k)$ the relations

$$ra_i^2\geqslant\theta_i-t, \tag{1.6}$$

$$(\theta_i-t)(\theta_k-t)\geqslant ra_i^2(\theta_k-t)+ra_k^2(\theta_i-t) \qquad (i\neq k). \tag{1.7}$$

We now distinguish several cases.

Case 1. If $t > \theta_1$, then (1.5) is equivalent with

$$1 + r\sum_{i\in S}\frac{a_i^2}{t-\theta_i} \geqslant 0 \qquad \text{for all} \quad S \subseteq \{1,\dots,v\}.$$

Since the left-hand side is positive when $r \geqslant 0$ and minimal for $S = \{1,\dots,v\}$ [with value $1 + rh_t(\Gamma)$] when $r < 0$, (1.5) is equivalent with $r \geqslant 0$ or $r < 0$ and $h_t(\Gamma) \leqslant -1/r$.

Case 2. If $t = \theta_1$, then (1.6) for $i = 1$ implies $r \geqslant 0$, since $a_1 = x^{(1)T}j > 0$ by Perron-Frobenius theory. In this case (1.5) is trivial, so that (1.5) is equivalent with $r \geqslant 0$.

Case 3. If $\theta_2 \leqslant t < \theta_1$, then (1.5) is violated unless $r > 0$ [apply (1.6) for $i = 1$] and $ra_i^2 = 0$ whenever $\theta_i = t$ [apply (1.6) and (1.7) with $k = 1$]. In this case (1.5) holds trivially if $1 \notin S$ or if S contains an i with $\theta_i = t$; hence (1.5) is equivalent with

$$1 + r\sum_{\substack{i\in S\\ a_i\neq 0}}\frac{a_i^2}{t-\theta_i} \leqslant 0 \qquad \text{if} \quad 1 \in S \subseteq \{1,\dots,v\}.$$

Since the left-hand side is maximal when $S = \{1,\dots,v\}$ and then has the value $1 + rh_t(\Gamma)$ (1.5) is equivalent with $h_t(\Gamma) \leqslant -1/r$.

Case 4. If $t < \theta_2$, then (1.5) is violated, since (1.6) and (1.7) imply $(\theta_1 - t)(\theta_2 - t) \geqslant (\theta_1 - t)(\theta_2 - t) + (\theta_2 - t)(\theta_1 - t)$, a contradiction. This implies the theorem. ∎

We now consider some simple cases where the height can be found explicitly.

Proposition 1.5. *Let Γ be a regular graph with v vertices and valency k. Then*

$$h_t(\Gamma) = \begin{cases} \dfrac{v}{t-k} & \text{if} \quad t \neq k, \\ \infty & \text{if} \quad t = k. \end{cases}$$

Proof. $x = (t-k)^{-1}j$ satisfies $(tI - A_\Gamma)x = j$. ∎

Proposition 1.6. *Let Γ_1, Γ_2 be graphs with finite t-height. Then their disjoint union $\Gamma_1 + \Gamma_2$ has finite t-height*

$$h_t(\Gamma_1 + \Gamma_2) = h_t(\Gamma_1) + h_t(\Gamma_2).$$

Proof. If $(tI - A_{\Gamma_i})x^{(i)} = j$ $(i = 1,2)$, then

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$$

satisfies $(tI - A_{\Gamma_1+\Gamma_2})x = j$. ∎

The *complete union* of two graphs Γ_1, Γ_2 (with disjoint vertex sets) is the graph $\Gamma_1 \oplus \Gamma_2$ obtained by adding to the vertices and edges of Γ_1 and Γ_2 all pairs $\{\gamma_1, \gamma_2\}$ with $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ as further edges. Its height can be neatly expressed using the transformation

$$h_t^*(\Gamma) := \frac{h_t(\Gamma)}{1 + h_t(\Gamma)}, \qquad h_t(\Gamma) = \frac{h_t^*(\Gamma)}{1 - h_t^*(\Gamma)} \tag{1.8}$$

on the closed real line $\mathbb{R} \cup \{\infty\}$.

Proposition 1.7. *Let Γ_1 and Γ_2 be graphs with finite height. Then their complete union $\Gamma_1 \oplus \Gamma_2$ satisfies*

$$h_t(\Gamma_1 \oplus \Gamma_2) = \frac{h_t(\Gamma_1) + h_t(\Gamma_2) + 2h_t(\Gamma_1)h_t(\Gamma_2)}{1 - h_t(\Gamma_1)h_t(\Gamma_2)},$$

$$h_t^*(\Gamma_1 \oplus \Gamma_2) = h_t^*(\Gamma_1) + h_t^*(\Gamma_2).$$

Proof. Let $\Gamma = \Gamma_1 \oplus \Gamma_2$; then

$$A_\Gamma = \begin{pmatrix} A_{\Gamma_1} & J \\ J^T & A_{\Gamma_2} \end{pmatrix}.$$

Hence if $(tI - A_{\Gamma_i})x^{(i)} = j$, $j^T x^{(i)} = h_t(\Gamma_i) = h_i$ $(i = 1,2)$, then

$$x = \frac{1}{1 - h_1 h_2}\begin{pmatrix} (1+h_2)x^{(1)} \\ (1+h_1)x^{(2)} \end{pmatrix}$$

satisfies $(tI - A_\Gamma)x = j$. Therefore $h_t(\Gamma) = (h_1 + h_2 + 2h_1h_2)/(1 - h_1h_2)$, which implies

$$\frac{h_t(\Gamma)}{1+h_t(\Gamma)} = \frac{h_1}{1+h_1} + \frac{h_2}{1+h_2}. \qquad \blacksquare$$

The preceding formulae allow now the computation of the heights of the complete graph K_n with n vertices, the complete bipartite graph $K_{m,n}$ with classes of size m, n, the complete multipartite graph $K_{s\times n}$ with s classes of size n, the circuit C_n with n vertices, and the disjoint union $n\Gamma$ of n copies of a graph Γ.

PROPOSITION 1.8.

$$h_t(mK_n) = \frac{mn}{t+1-n}, \qquad h_t(C_n) = \frac{n}{t-2},$$

$$h_t(K_{m,n}) = \frac{2mn + t(m+n)}{t^2 - mn}, \qquad h_t(K_{s\times n}) = \frac{sn}{t-(s-1)n}.$$

Proof. For $\Gamma = K_n$ and C_n apply Proposition 1.5. Then $h_t(m\Gamma) = mh_t(\Gamma)$ follows from Proposition 1.6, and Proposition 1.7 gives $h_t^*(K_{m,n}) = h_t^*(mK_1) + h_t^*(nK_1)$, $h_t^*(K_{s\times n}) = sh_t^*(nK_1)$. Using the transformation (1.8), we thus get

$$h_t^*(nK_1) = \frac{n}{t+n}, \qquad h_t^*(K_{m,n}) = \frac{2mn + t(m+n)}{(t+m)(t+n)}, \qquad h_t^*(K_{s\times n}) = \frac{sn}{t+n},$$

and from this we get the stated formulae. $\blacksquare$

This has an immediate application.

COROLLARY 1.9. *Let Γ be a graph with an $(r+t, r-1, r)$-representation, $t > 0$.*

(i) *If $r < 1$, then the size of an n-clique K_n is bounded by $n \leqslant (t+1)/(1-r)$.*
(ii) *If $r < 0$, then the size of an n-coclique nK_1 is bounded by $n \leqslant -t/r$.*

(iii) *If $r < 1/(t+2)$, then the size of an n-claw $K_{1,n}$ is bounded by*

$$n \leqslant \frac{t(r+t)}{1-(t+2)r}.$$

Proof. Apply Proposition 1.2 with the values of Proposition 1.8. $\blacksquare$

Let Δ be a subgraph of a graph Γ, and let a be a $(0,1)$-vector indexed by Δ. Inspired by Lemmens and Seidel [8], we define the *a-pillar* of Δ (in Γ) as the subgraph induced on the set

$$P_a(\Delta) := \{\gamma \in \Gamma \setminus \Delta \mid \gamma \sim \delta \text{ if } a_\delta = 1,\ \gamma \nsim \delta \text{ if } a_\delta = 0\ (\delta \in \Delta)\}.$$

Suppose that Γ has a $(r+t, r-1, r)$-representation, and let $P = P_a(\Delta)$ be a pillar of Γ. Then the Gram matrix of the image of $\Delta \cup P$ can be written in the form

$$G_0 = \begin{pmatrix} G_\Delta & (rj-a)j^T \\ j(rj-a)^T & G_p \end{pmatrix}.$$

(Here j and j^T are all-one vectors of size $|P|$ and $|\Delta|$, respectively.) If the image of Δ is linearly independent, then $G_\Delta = tI - A_\Delta + rJ$ is nonsingular, and the Schur complement G_p' of G_0 with respect to G_Δ can be formed. We find $G_p' = tI - A_p + r_a(\Delta)J$, where

$$r_a(\Delta) = r - (rj-a)^T(tI - A_\Delta + rJ)^{-1}(rj-a) < r. \tag{1.9}$$

Since Schur complements of positive semidefinite matrices are again positive semidefinite, we have:

PROPOSITION 1.10. *Let Γ have a $(r+t, r-1, r)$-representation and let Δ be a subgraph of Γ with linearly independent image. Then the a-pillar of Δ has an (r_a+t, r_a-1, r_a)-representation, where $r_a = r_a(\Delta)$ is given by (1.9). In particular, if $B := tI - A_\Delta$ is nonsingular, then*

$$r_a(\Delta) = \frac{r(1+j^TB^{-1}a)^2}{1+rj^TB^{-1}j} - a^TB^{-1}a. \tag{1.10}$$

Proof. It only remains to verify (1.10). For an arbitrary vector b one easily verifies that $x = B^{-1}b + \alpha B^{-1}j$ is a solution of the equation $(B + rJ)x = b$ when $\alpha = -rj^TB^{-1}b/(1 + rj^TB^{-1}j)$.

Multiplication with b^T gives the basic relation

$$b^T(B + rJ)^{-1}b = b^TB^{-1}b - \frac{r(j^TB^{-1}b)^2}{1 + rj^TB^{-1}j}, \tag{1.11}$$

since by symmetry of B we have $b^TB^{-1}j = j^TB^{-1}b$. We now specialize to $b = rj - a$. With the abbreviations $u_0 = j^TB^{-1}j$, $u_1 = j^TB^{-1}a$, $u_2 = a^TB^{-1}a$, the right-hand side of (1.11) becomes

$$r^2u_0 - 2ru_1 + u_2 - \frac{r(ru_0 - u_1)^2}{1 + ru_0} = r + u_2 - \frac{r(1 + u_1)^2}{1 + ru_0},$$

and (1.10) follows from this together with (1.9). ■

This result has the following relevance for graphs with an $(r + t, r - 1, r)$-representation. If $r_a(\Delta) \leqslant 0$, then the a-pillar $P_a(\Delta)$ has a $(t, -1, 0)$-representation; hence, by Proposition 1, it is a graph with largest eigenvalue $\leqslant t$. For $t \leqslant 2$ (or slightly larger [5]), all these graphs are known (Smith [15]); thus for certain values of r and certain choices of Δ, a substantial part of Γ has a known structure. We show in the next section that this can sometimes be exploited to determine the structure of Γ itself.

Similarly, if $r_a(\Delta) \leqslant 1$, then the a-pillar $P_a(\Delta)$ of Δ has a $(t + 1, 0, 1)$-representation; hence, by Proposition 1, it is the complement of a graph with smallest eigenvalue $\geqslant -1 - t$ Thus, for $t \leqslant 1$ (smallest eigenvalue $\geqslant -2$), the characterization theorems of Cameron et al. [3] and Bussemaker et al. [2] can be applied to $P_a(\Delta)$.

If the image of Δ is linearly dependent, then each dependency relation gives a restriction on the possible nonempty a-pillars of Δ:

Proposition 1.11. *Let Δ be a subgraph of a graph Γ with an $(r + t, r - 1, r)$-representation. Then, for any vector c indexed by Δ, the equation*

$$\sum_{\delta \in \Delta} c_\delta \bar{\delta} = 0 \tag{1.12}$$

holds iff one of the following conditions is satisfied:

(i) $h_t(\Delta) = -1/r$ *and* $(tI - A_\Delta)c \| j$,
(ii) $(tI - A_\Delta)c = 0$, $r(j^Tc) = 0$.

Moreover, for any c satisfying (1.12) *and any nonempty pillar $P_a(\Delta)$, we have*

$$\sum_{\delta \in \Delta} a_\delta c_\delta = r \sum_{\delta \in \Delta} c_\delta. \tag{1.13}$$

Proof. (1.12) is equivalent with $G_\Delta c = 0$; indeed, (1.12) implies $0 = \Sigma(\bar{\gamma}, \bar{\delta})c_\delta = (G_\Delta c)_\gamma$ for each $\gamma \in \Delta$, hence $G_\Delta c = 0$; and conversely, if this holds, then $0 = c^TG_\Delta c = (\Sigma c_\delta \bar{\delta}, \Sigma c_\delta \bar{\delta})$, so that (1.12) holds. Now $G_\Delta = tI - A_\Delta + rJ$, and since $J = jj^T$, we have $G_\Delta c = 0$ iff

$$(tI - A_\Delta)c = -r(j^Tc)j. \tag{1.14}$$

Clearly (1.14) holds in case (ii). Hence suppose that $r(j^Tc) \neq 0$. Then (1.14) implies $(tI - A_\Delta)c \| j$, and $x = -c/r(j^Tc)$ satisfies $(tI - A_\Delta)x = j$; hence $h_t(\Delta) = j^Tx = -1/r$. Therefore (i) holds; conversely, (i) implies (1.14) and hence (1.12).

Finally, if $\gamma \in P_a(\Delta)$, then (1.12) implies $0 = \Sigma(\bar{\gamma}, \bar{\delta})c_\delta = \Sigma(r - a_\delta)c_\delta$, which gives (1.13). ■

It would be useful to have also in this case a result similar to Proposition 1.10.

2. EQUIANGULAR LINES

We now apply the preceding general results to the special case $r = \frac{1}{2}$.

Proposition 2.1. *A graph has a $(t + \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation iff the smallest eigenvalue $\sigma_{\min}(\Gamma)$ of the matrix $C_\Gamma = J - I - 2A_\Gamma$ satisfies $\sigma_{\min}(\Gamma) \geqslant -2t - 1$.*

Proof. Γ has a $(t + \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation iff $G_\Gamma = tI - A_\Gamma + \frac{1}{2}J$ is positive semidefinite. Since $2G = C_\Gamma + (2t + 1)I$, this is equivalent with $\sigma_{\min}(\Gamma) \geqslant -2t - 1$. ■

The matrix C_Γ is the $(0, -1, 1)$-adjacency matrix of the graph Γ, introduced by Seidel [11] in the context of *equiangular lines*; cf. also [9, 8, 12]. Indeed, the image of Γ under a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation consists of vectors of norm $t+\frac{1}{2}$, and the lines through these vectors have constant mutual angle $\arccos[1/(2t+1)]$; conversely, if X is a set of equiangular lines with constant mutual angle $\arccos c$, and if we pick along each line a vector of length $(2c)^{-1/2}$, then any two of these vectors have inner product $1/c \pm \frac{1}{2}$, thus giving a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation, where $t=(1-c)/2c$.

The fact that each line contains two opposite vectors of the same length allows one to modify a graph Γ in certain ways without changing the set of equiangular lines corresponding to a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation of Γ. Replacing certain vectors by there opposites amounts to multiplication of the corresponding rows and columns of c_Γ by -1; in terms of adjacencies we obtain from Γ a graph Γ_S by *switching* with respect to a subset S of vertices, i.e. by reversing the adjacencies between vertices in S and vertices not in S (Seidel [11]). Thus the graph Γ_S has the same vertex set as Γ, and $\{\gamma, \delta\}$ is an edge in Γ_S iff either $|\{\gamma,\delta\} \cap S| \neq 1$ and $\{\gamma,\delta\}$ is an edge of Γ or $|\{\gamma,\delta\}\cap S| = 1$ and $\{\gamma,\delta\}$ is not an edge of Γ. We call two graphs Γ, Γ' *switching equivalent* if Γ' can be obtained from Γ by switching at a suitable set S. Since switching at S followed by switching at S′ is equivalent to switching at their symmetric difference, this defines indeed an equivalence relation.

Clearly, switching equivalent graphs determine isomorphic sets of equiangular lines, and with a graph Γ, every graph switching equivalent to Γ has a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation with the same value of t. In particular, we have:

Proposition 2.2. *Let Γ be a graph which is switching equivalent to a graph with largest eigenvalue $\leqslant t$. Then Γ has a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation.*

Proof. By switching, we may assume that $\theta_{\max}(\Gamma) \leqslant t$. Then $tI - A_\Gamma$ and hence $tI - A_\Gamma + \frac{1}{2}$ are positive semidefinite. ∎

For $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representations, the possibility of switching provides a method for the calculation of the values $r_a(\Delta)$ for pillars of subgraphs which also works when the image of Δ is dependent.

Proposition 2.3. *Let Δ be a subgraph of a graph Γ with a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation, let $P_a(\Delta)$ be a nonempty pillar of Δ, and let Δ_a be the graph obtained by switching Δ with respect to $S = \{\delta \in \Delta \mid a_\delta = 1\}$. Then $h_t(\Delta_a) \neq -2$, and $P_a(\Delta)$ has an (r_a+t, r_a-1, r_a)-representation, where*

$$r_a = r_a(\Delta) = [2 + h_t(\Delta_a)]^{-1}. \tag{2.1}$$

Proof. The graph Γ_a obtained by switching Γ with respect to S has a $(t+\frac{1}{2}, +\frac{1}{2}, \frac{1}{2})$-representation, and $P_a(\Delta)$ is the 0-pillar of Δ_a in Γ_a. In particular, $\Delta_a + P_a(\Delta)$ has a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation. Suppose first that

$$h_t(\Delta_a) + h_t(P_a(\Delta)) \leqslant -2. \tag{2.2}$$

If $h_t(\Delta_a) = -2$ then $h_t(P_a(\Delta)) \leqslant 0$; hence by Propositions 1.2 and 1.3, both Δ_a and $P_a(\Delta)$ have largest eigenvalue $>t$, so that $\theta_2(\Delta_a + P_a(\Delta)) > t$, contradicting Theorem 1.4. Hence $h_t(\Delta_a) \neq -2$, and with r_a defined by (1) we have $h_t(P_a(\Delta)) \leqslant -1/r_a$, so that the assertion follows from Theorem 1.4. Indeed, we have to check that either $r_a < 0$ and $\theta_1(P_a(\Delta)) < t$ or $r_a > 0$ and $\theta_2(P_a(\Delta)) \leqslant t$. But we know that $\theta_2(\Delta_a + P_a(\Delta)) \leqslant t$; hence it suffices to exclude the situation $r_a < 0$, $\theta_1(P_a(\Delta)) > t$ [equality is impossible, otherwise $h_t(P_a(\Delta)) = +\infty$]. But if $r_a < 0$, then by Proposition 1.3 we have $\theta_1(\Delta) > t$, contradicting $\theta_2(\Delta_a + P_a(\Delta)) \leqslant t$.

Now suppose that (2.2) is wrong. Then Theorem 1.4 shows that the largest eigenvalue of $\Delta_a + P_a(\Delta)$ and hence that of $P_a(\Delta)$ is $\leqslant t$; in this case, the assertion even holds for arbitrary $r_a > 0$. ∎

Remark. If t is not an eigenvalue of Δ_a, a shorter proof is obtained by applying Proposition 1.10 with $a = 0$ to Δ_a in place of Δ.

Proposition 2.4. *Let Γ be a graph with a $(t+\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation, $t \geqslant 1$. Then:*

(i) *The size of an n-clique is bounded by $n \leqslant 2t+2$.*

(ii) *If C is a coclique and $\gamma \notin C$, then the number p of neighbors of γ in C and the number q of nonneighbors of γ in C are related by*

$$(p-t^2)(q-t^2) \leqslant t^2(t+1)^2. \tag{2.3}$$

In particular,

$$\min(p,q) \leqslant 2t^2 + t. \tag{2.4}$$

Proof. (i) is a special case of Corollary 1.9(i). To prove (ii) we note that the graph induced on $C \cup \{\gamma\}$ is $K_{1,p}+qK_1$ with height

$$\frac{2p+t(p+1)}{t^2-p}+\frac{q}{t}=\frac{t(t+1)^2}{t^2-p}+\frac{q-t^2}{t}-2.$$

Since the height is $\leqslant -1/r \leqslant -2$ by Theorem 1.4, we get (2.3), unless $\sqrt{p}=\theta_1(K_{1,p})=\theta_1(K_{1,p}+qK_1)\leqslant t$; but in this case (2.3) also holds. Clearly, (2.3) implies (2.4). ■

COROLLARY 2.5. *Let Γ be a graph with a $(t+\frac{1}{2},\,-\frac{1}{2},\frac{1}{2})$-representation, and let C be a coclique in Γ. Then the vertices of Γ not in C can be switched so that they have $\leqslant 2t^2+t$ neighbors in C.* ■

As an application we prove a kind of converse to Proposition 2.2. For the statement of the result we need the following concept. A set $G^{\#}$ of graphs is called a *complete list of minimal forbidden subgraphs* for a class G of graphs if $G^{\#}$ consists of pairwise nonisomorphic graphs $\notin G$ such that every proper subgraph of $G^{\#}$ is in G, and every graph not in G contains a subgraph isomorphic to a graph in $G^{\#}$. Clearly, $G^{\#}$ is determined by G up to isomorphism. We need $G_t^{\#}$ for the class G_t of graphs with largest eigenvalue $\leqslant t$.

THEOREM 2.6. *If $G_t^{\#}$ is finite, then there exists a number $v(t)$ such that every graph with more than $v(t)$ vertices having a $(t+\frac{1}{2},\,-\frac{1}{2},\frac{1}{2})$-representation is switching equivalent to a graph with largest eigenvalue $\leqslant t$.*

Proof. Let m be the maximal number of vertices of graphs in $G_t^{\#}$, and let h be the minimal value of the t-height of graphs in $G_t^{\#}$. Put $s_1=\lfloor 2t+2\rfloor$, $s_2=\lfloor m(2t^2+t+1)-t(h+2)\rfloor$, and let $v(t)$ be the Ramsey number $R(s_1+1,s_2+1)$, so that every graph Γ with more than $v(t)$ vertices contains either an (s_1+1)-clique or an (s_2+1)-coclique. If Γ has an $(t+\frac{1}{2},\,-\frac{1}{2},\frac{1}{2})$-representation, then the first possibility is impossible by Proposition 2.4(i), so that Γ has an (s_2+1)-coclique C. By Corollary 2.5 we may assume w.l.o.g. that all vertices of Γ have at most $2t^2+t$ neighbors in C.

Now suppose that $\theta_{\max}(\Gamma)>t$. Then Γ contains a subgraph Δ isomorphic to one of the graphs in $G_t^{\#}$; in particular, Δ contains at most m vertices, and $h_t(\Delta)\geqslant h$. Now C contains at most $m(2t^2+t+1)$ vertices at distance $\leqslant 1$ from Δ; hence the 0-pillar of Δ contains a coclique C_0 of size $\geqslant s_2+1-m(2t^2+t+1)>-t(h+2)$ by construction of s_2. As a subgraph of Γ, the graph $\Delta+C_0$ has a $(t+\frac{1}{2},\,-\frac{1}{2},\frac{1}{2})$-representation, and by Theorem 1.4 we have $h_t(\Delta+C_0)\leqslant -2$. But by Propositions 1.6 and 1.8 we have $h_t(\Delta+C_0)=h_t(\Delta)+|C_0|/t>h-(h+2)=-2$, contradiction. Hence $\theta_{\max}(\Gamma)\leqslant t$. ■

Let v_t (possibly ∞) denote the maximal number of vertices of a graph Γ with a $(t+\frac{1}{2},\,-\frac{1}{2},\frac{1}{2})$-representation such that Γ is not switching equivalent to a graph with largest eigenvalue $\leqslant t$. Results implicitly in Lemmens and Seidel [8] and Shult and Yanushka [14] imply that $v_2=28$, and Theorem 2.6 shows that $v_t<\infty$ if $G_t^{\#}$ is finite.

In particular, since for $1<t<\sqrt{2}$ the set $G_t^{\#}$ consists of the 2-claw (with largest eigenvalue $\sqrt{2}$) and the triangle (with largest eigenvalue 2) only, we have $v_t<\infty$ for $1<t<\sqrt{2}$. In other words, every sufficiently large graph with a $(t+\frac{1}{2},\,-\frac{1}{2},\frac{1}{2})$-representation with fixed $t<\sqrt{2}$ is switching equivalent to a graph consisting of isolated vertices and edges.

Lists compiled by Koszul [6] and Chein [4] show that $G_2^{\#}$ is a finite set consisting of 18 graphs. Thus the hypothesis of the theorem is satisfied for $t=2$, and hence $v_2<\infty$. In particular, since Smith [15] determined all graphs with largest eigenvalue $\leqslant 2$, this implies that all sufficiently large sets of equiangular lines with angle $\arccos\frac{1}{5}$ are known. This complements the results of Lemmens and Seidel [8]. By sharpening the arguments used in the proof of Theorem 2.6, and using the explicit knowledge of $G_2^{\#}$, reasonably small explicit bounds for v_3 can be obtained ($2486\leqslant v_3\leqslant 45374$). This will be done in a separate paper.

Finally, we remark that the recent result of Shearer [13], that every number $t\geqslant t^*=(2+\sqrt{5})^{1/2}\approx 2.058$ is a limit point from above of the set of largest eigenvalues of graphs, makes it likely that the hypothesis of Theorem 2.6 can be satisfied if and only if $t<t^*$. (As communicated to me by Professor J. J. Seidel, Eindhoven, this has indeed been verified by A. J. Hoffman and J. Shearer.) Thus the next interesting case, $t=3$, will require substantially stronger techniques.

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