Surprisingly, the inner solution is valid even in the outer region. Moreover, this inner solution is the singular perturbation solution ($u_{sp}$). That is,

$$u_{sp} = u^i = e^{-(1/\varepsilon)y}. \tag{13}$$

On comparing (12) and (13), we see that the effect of magnetic field is almost negligible on the flow field.

## 4. Solution and discussion for small suction Reynolds number

For small $R_0$, equation (6) and conditions (7) can be written as

$$\frac{d^2u}{dy^2} + \varepsilon \frac{du}{dy} - \varepsilon Mu = 0, \tag{14}$$

$$\left.\begin{array}{lll} u = 1 & \text{at} & y = 0, \\ u \to 0 & \text{as} & y \to \infty, \end{array}\right\} \tag{15}$$

where $\varepsilon = R_0$.

For equations (14) and (15) a straightforward perturbation expansion is not possible because of the infinity condition. However, the exact solution in this case is

$$u = e^{-(1/2)[\varepsilon + (\varepsilon^2 + 4\varepsilon M)^{1/2}]y}. \tag{16}$$

From (16), it is evident that the velocity field $u$ decreases with increasing Hartmann number. This phenomenon is true even with the suction Reynolds number.

## References


1 SAKIADIS, B. C.: Boundary-layer behavior on continuous solid surfaces. A. I. Ch. E. J. 7 (1961), 26; 7 (1961), 221.
2 BLASIUS, H.: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Z. Math. Phys. 56 (1908), 1.
3 TSOU, F.; SPARROW, E. M.; GOLDSTEIN, R.: Flow and heat transfer in the boundary layer on a continuous moving surface. Int. J. Heat Mass Transfer 10 (1967), 219.
4 ABDELHAFEZ, T. A.: Skin friction and heat transfer on a continuous flat surface moving in a parallel free stream. Int. J. Heat Mass Transfer 28 (1985), 1234.
5 SHERCLIFF, J. A.: A text book of magnetohydrodynamics. Pergamon Press 1965.
6 MEREDITH, F. M.; GRIFFITH, A. A.: Modern developments in fluid dynamics. Oxford University Press 1938.
7 SCHLICHTING, H.: Die Grenzschicht mit Absaugung und Ausblasen. Luftfahrtforschung 19 (1942), 179.


Received June 13, 1987

*Address*: Prof. Dr. K. VAJRAVELU, Department of Mathematics, University of Central Florida, Orlando, FL 32816, U.S.A.

ZAMM · Z. angew. Math. Mech. 68 (1988) 6, 256–257

NEUMAIER, A.

## An Existence Test for Root Clusters and Multiple Roots

There is a well-known existence and enclosure test for roots of odd multiplicity of a continuous real function $f(x)$ in a single unknown: If $f(x_1) f(x_2) \leqq 0$ then $f$ has a root between $x_1$ and $x_2$. This simple test is very powerful, and is the basis of most globally convergent root finders (see, e.g. the comparison in NERINCKX and HAEGEMANS [3]). In the context of interval analysis, some other existence tests, based on Brouwer's fixpoint theorem, are in use (see e.g. RUMP [6]); for real functions, these tests are less powerful than sign tests, but for simple roots of complex analytic functions they are quite convenient. Both the sign change test and the known interval tests become more and more difficult to apply when two roots are very close, and they fail for double roots. Indeed, there cannot be a general purpose existence test for double roots (and roots of even multiplicity) of real functions since such roots disappear when arbitrarily small perturbations of suitable sign are applied to $f$.

On the other hand, under small perturbations of a complex analytic function, a double root remains double or splits into two simple roots. Thus the problem of deciding whether a double root or two simple roots are very close to a given number $\tilde{z}$ is well-posed. More generally, the number $n(f, D)$ of roots in $D$ of a function $f$ which is analytic in an open, bounded set $D \subseteq \mathbb{C}$ and continuous and nonzero on $\partial D$ is invariant under small perturbations of $f$, if each root is counted according to its multiplicity. This is a particular case of the homotopy invariance of $n(f, D)$, which we state here as follows.

Homotopy Invariance Theorem: *Let $E$ be a connected Hausdorff space, and let $D_0$ be a subset of $\mathbb{C}$. Suppose that $g: D_0 \times E \to \mathbb{C}$ is a continuous function such that each function $g_t (t \in E)$ defined by $g_t(z) = g(z, t)$ for $z \in D_0$ is analytic in some open and bounded set $D$ with $\overline{D} \subseteq D_0$, and nonzero on $\partial D$. Then $n(g_t, D)$ is independent of $t \in E$.*

Proof: If $D$ is simply connected and $\partial D$ is a (positively oriented) Jordan curve then

$$n(f, D) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz \tag{1}$$

(HENRICI [2], § 4.10); thus $n(g_t, D)$ is an integer depending continuously on $t$, and is therefore constant. The general case follows from the homotopy invariance theorem of Leray-Schauder degree theory (cf. ORTEGA and RHEINBOLDT [4], Ch. 6) since $n(f, D)$ is the degree $d(F, D, 0)$ of

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \operatorname{re} f(u + iv) \\ \operatorname{im} f(u + iv) \end{pmatrix};$$

this follows from $d(F, D, 0) = \sum \operatorname{sgn} (\det F'(x))$ summed over all $x \in D$ with $F(x) = 0$, and

$$\det F'\begin{pmatrix} u \\ v \end{pmatrix} = \det \begin{pmatrix} \operatorname{re} f' & -\operatorname{im} f' \\ \operatorname{im} f' & \operatorname{re} f' \end{pmatrix} = |f'|^2 \geqq 0. \ \square$$

Any rule for evaluation of $n(f, D)$ amounts to an existence and enclosure test $n(f, D) \neq 0$ for one or several roots of $f$ in $D$. That the number $n(f, D)$ can be expressed explicitely in terms of the integral (1) has occasionally been used for numerical root location (see HENRICI [2], § 6.11). However, to rigorously ascertain the accuracy of a numerical integration is a highly nontrivial matter. In the following we therefore derive a simple and practically sufficient condition for checking whether $n(f, D) = k$ for a given integer $k$. This test can be implemented rigorously on any computer which supports rounded interval arithmetic.

Proposition: *Let $f: D_0 \to \mathbb{C}$ be analytic in an open and bounded set $D$ with $\overline{D} \subseteq D_0$, and continuous on $\partial D$. If $\tilde{z} \in D$ and*

$$\operatorname{Re} \frac{f(z)}{(z - \tilde{z})^k} > 0 \quad \textit{for all} \quad z \in \partial D \tag{2}$$

*then $n(f, D) = k$.*

Proof: Define $g: D_0 \times [0, 1] \to \mathbb{C}$ by $g(z, t) := tf(z) + (1 - t)(z - \tilde{z})^k$. Then $g_0(z) = (z - \tilde{z})^k$ and $g_1(z) = f(z)$. Now $g_t(z) \neq 0$ for $t \in [0, 1]$ and $z \in \partial D$; this is trivial for $t = 0$ and follows for $t \in [0, 1]$ since

$$\operatorname{Re} \frac{g_t(z)}{(z - \tilde{z})^k} = t \operatorname{Re} \frac{f(z)}{(z - \tilde{z})^k} + 1 - t > 1 - t \geqq 0.$$

Hence, by the homotopy invariance theorem, $n(f, D) = n(g_1, D) = n(g_0, D) = k$. $\square$

We now apply this proposition to circular discs

$$K(\tilde{z}; r) = \{z \in \mathbb{C} \mid |z - \tilde{z}| \leqq r\}. \tag{3}$$

Theorem: *Let $f$ be analytic in the interior of $K(\tilde{z}; r)$, and let $0 < \varepsilon < r$. If*

$$\left| \operatorname{Re} \frac{f^{(k)}(z)}{k!} \right| > \sum_{i=0}^{k-1} \left| \frac{f^{(i)}(\tilde{z})}{i!} \right| \varepsilon^{i-k} \quad \textit{for all} \quad z \in K(\tilde{z}; \varepsilon) \tag{4}$$

*then $f$ has precisely $k$ roots in $K(\tilde{z}; \varepsilon)$, where each root is counted according to its multiplicity.*

Proof: Put $D = \operatorname{int} K(\tilde{z}; \varepsilon)$, and denote the right hand side of (4) by $\sigma$. By continuity, $\operatorname{Re} f^{(k)}(z)$ has constant sign on $K(\tilde{z}; \varepsilon)$, and without loss of generality we may take it positive, so that in particular

$$\operatorname{Re}\frac{f^{(k)}(z)}{k!} > \sigma \quad \text{for all} \quad z \in \overline{D}\,.$$

Now Taylor's theorem implies

$$-\operatorname{Re}\frac{f(z)}{(z-\tilde{z})^k} = -\operatorname{Re}\left(\sum_{i<k}\frac{f^{(i)}(\tilde{z})}{i!}(z-\tilde{z})^{i-k} + \int_0^1 \frac{f^{(k)}(t\tilde{z}+(1-t)z)}{(k-1)!}t^{k-1}\,\mathrm{d}t\right) \leqq$$

$$\leqq \sum_{i<k}\left|\frac{f^{(i)}(\tilde{z})}{i!}\right| |z-\tilde{z}|^{i-k} - \int_0^1 \operatorname{Re}\frac{f^{(k)}(t\tilde{z}+(1-t)z)}{(k-1)!}t^{k-1}\,\mathrm{d}t\,.$$

For $z \in \partial D$, the sum equals $\sigma = \int_0^1 \sigma k t^{k-1}\,\mathrm{d}t$; hence

$$-\operatorname{Re}\frac{f(z)}{(z-\tilde{z})^k} \leqq \int_0^1 \left(\sigma - \operatorname{Re}\frac{f^{(k)}(t\tilde{z}+(1-t)z)}{k!}\right)kt^{k-1}\,\mathrm{d}t \leqq 0\,,$$

and the proposition applies. □

Remarks:

1. With (rounded) complex interval arithmetic (see Henrici [2], § 6.6 for circle arithmetic, Alefeld and Herzberger [1] for rectangle arithmetic) it is easy to find an enclosure for the numbers $\operatorname{Re} f^{(k)}(z)/k!$ with $z \in K(z;\varepsilon)$ (or, in rectangl earithmetic, with $z$ in the square enclosing $K(z,\varepsilon)$) defined by

$$\operatorname{Re}(z-\tilde{z}),\ \operatorname{Im}(z-\tilde{z}) \in [-\varepsilon,\varepsilon]$$

once an arithmetical expression for $f$ is known. The higher derivatives can be computed automatically and efficiently by simple recursions, see Rall [5].

2. In practice, the theorem is best used as an aposteriori test for existence, starting with an approximate root $\tilde{z}$ computed by standard numerical methods. The theorem then provides a rigorous existence test, multiplicity count, and enclosure for the root or the root cluster near $\tilde{z}$. The successful application requires that we "guess" the right $k$ and a suitable $\varepsilon$. Since $\varepsilon$ should be kept small, a reasonable procedure is the following: Let $\varepsilon_k$ be the positive real root of

$$\left|\frac{f^{(k)}(\tilde{z})}{k!}\right|\varepsilon^k - \sum_{i=1}^{k-1}\left|\frac{f^{(i)}(\tilde{z})}{i!}\right|\varepsilon^i\,;$$

then (4) forces $\varepsilon > \varepsilon_k$. If $\varepsilon_k$ is small then $f^{(k)}(z) = f^{(k)}(\tilde{z}) + O(\varepsilon)$ so that it is sufficient to take $\varepsilon$ only slightly bigger than $\varepsilon_k$. In practice it is usually sufficient to calculate $\varepsilon_k$ to a relative precision of 10 percent only and to choose $\varepsilon = 2\varepsilon_k$; since $k$ is unknown one tries $k = 1, 2, \ldots$, until one succeeds or a limit on $k$ is attained.

3. If $k$ roots are at distance $\lesssim \bar{\varepsilon}$ from $\tilde{z}$ and the remaining roots are far away then we have roughly

$$\frac{f^{(i)}(\tilde{z})}{i!} \approx \binom{k}{i}\frac{f^{(k)}(\tilde{z})}{k!}(\bar{\varepsilon})^{k-i}\,.$$

This shows that if $f^{(k)}(\tilde{z})$ is not real, we must divide by $f^{(k)}(\tilde{z})$ (or a real multiple of it) before applying the theorem. If we do this we find that

$$\varepsilon_k \approx \bar{\varepsilon}/(\sqrt[k]{2}-1) < 1.5k\bar{\varepsilon}\,,$$

so that we get with our choice an overestimation of roughly $3k$.

4. The Schur-Cohn algorithm (Henrici [2], § 6.8) calculates $n(f, D)$ for polynomials $f$ of degree $n$ and circles $D$ in $O(n^2)$ operations. In contrast, our sufficient criterion requires for polynomials only $O(n)$ operations, and can also be applied to nonpolynomial functions.

Example: The most important case is that of a double root (or two very close roots), i.e. $k = 2$. In this case, a suitable guess for the radius $\varepsilon$ is

$$\varepsilon = 2\varepsilon_2 = p + \sqrt{p^2 + 4q}\,,$$

where $p = |2f'(\tilde{z})/f''(\tilde{z})|$, $q = |2f(\tilde{z})/f''(\tilde{z})|$, and the condition guaranteeing for two roots in the circle $z = K(\tilde{z};\varepsilon)$ (to be verified e.g. in complex circle arithmetic) is

$$\inf\left(\operatorname{Re}\frac{f''(z)}{f''(\tilde{z})}\right) > (p + q/\varepsilon)/\varepsilon\,.$$

We apply this to $f(x) = x^4 - 2x^3 - x^2 + 2x + 1$ which has a double root $x^* = \frac{1}{2}(1+\sqrt{5}) \approx 1.618033989$, and obtain for various approximations the enclosures $|x^* - \tilde{z}| \leqq \varepsilon$ given in Table 1.

Table 1

| $\tilde{z}$ | $\varepsilon$ (upward rounded) | $\varepsilon/\lvert x^* - \tilde{z}\rvert$ |
|---|---|---|
| 1.5 | $6.77\ 10^{-1}$ | enclosure not guaranteed |
| 1.6 | $8.91\ 10^{-2}$ | 4.94 |
| 1.61 | $3.92\ 10^{-2}$ | 4.88 |
| 1.618 | $1.65\ 10^{-4}$ | 4.83 |
| 1.618034 | $5.44\ 10^{-8}$ | 4.83 |
| 1.62 | $9.48\ 10^{-3}$ | 4.82 |
| 1.65 | $1.49\ 10^{-1}$ | 4.66 |
| 1.7 | $3.60\ 10^{-1}$ | enclosure not guaranteed |

Of course the test doesn't guarantee for the existence of a double root but only for that of two (possibly coinciding) roots $x^*$ with $|x^* - \tilde{z}| \leqq \varepsilon$.

## References

1 Alefeld, G.; Herzberger, J.: Introduction to interval computations. Academic Press, New York 1983.
2 Henrici, P.: Applied and computational complex analysis. Vol. 1. Wiley, New York-London 1974.
3 Nerinckx, D.; Haegemans, A.: A comparison of non-linear equations solvers. J. Comput. Appl. Math. 2 (1976), 145–148.
4 Ortega, J. M.; Rheinboldt, W. C.: Iterative solution of non-linear equations in several variables. Academic Press 1970.
5 Rall, L. B.: Automatic differentiation: Techniques and applications. Springer Lecture Notes in Comp. Sci. 120, Berlin 1981.
6 Rump, S.: Solution of linear and nonlinear algebraic problems with sharp, guaranteed bounds. Computing Suppl. 5 (1984), 147–168.

Received May 15, 1987

*Address*: Dr. Arnold Neumaier, Albert-Ludwigs-Universität, Institut für Angewandte Mathematik, Hermann-Herder-Straße 10, D-7800 Freiburg i. Br., F.R.G.