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OVERESTIMATION IN LINEAR INTERVAL EQUATIONS*

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Abstract. Bounds are given for the overestimation of the solution set of a system of linear interval equations by various inclusion intervals for this set. In particular, a theorem of Gay [this Journal, 19 (1982), pp. 858–870] on the quadratic approximation property of the preconditioned fixpoint inverse is strengthened.

Key words. linear interval equations, overestimation, quadratic approximation

AMS(MOS) subject classifications. 65G10, 65F10

1. Introduction. A well-studied problem in interval analysis is the construction of enclosures for the set of solutions of the equations

$$\tilde A\tilde x=\tilde b \qquad (\tilde A\in A,\ \tilde b\in b), \tag{1}$$

where A is a regular interval matrix (i.e. all $\tilde A\in A$ are nonsingular) and b is an interval vector. While preconditioned Gauss elimination (Hansen [6]) and preconditioned interval iteration (Krawczyk [7]) turned out to be practically useful methods, the quality of the computed enclosures was paid little attention, apart from the special case of M-matrices. There, Barth and Nuding [2] showed that the limit A^Fb of the Jacobi iteration for (1) agrees with the hull A^Hb of the solution set of (1), and Barth and Nuding [2] and Beeck [3] showed the same for the result A^Gb of Gauss elimination provided that the right-hand side b satisfies $0\leqq b$, $0\geqq b$, or $0\in b$.

In the general case an asymptotic investigation of the overestimation of preconditioned Gauss elimination has been made by Miller [10] (cf. the exposition in Alefeld and Herzberger [1, Chap. 16]). Moreover, exponential worst case overestimation of Gauss elimination without preconditioning was demonstrated by Wongwises [15] and Schätzle [14] under natural simplifying assumptions. For the overestimation in case of preconditioned interval iteration a neat theorem of Gay [5] is available; however, his proof contains a gap. The present paper partly owes its existence to an attempt to repair the proof; in fact we prove a strengthened version of Gay's theorem (Theorem 3). We also prove a new optimality result for the midpoint inverse as preconditioning matrix (Theorem 4) and derive another enclosure for the hull A^Hb with small overestimation, based on an approximate inverse[1] C and an approximate solution $\check x$ (Theorem 1). Using the concept of a skew vector, we are even able to approximate the solution set of (1) itself by a skew vector with small overestimation (Theorem 2).

The terminology follows Neumaier [12]; it is assumed that the reader is acquainted with §§ 1–4 and 7 of [12]. However, for convenience of the reader we repeat some basic notation. We denote by $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times n}$ the set of real numbers, n-dimensional column vectors and $n\times n$-matrices, and by $\mathbb{IR}$, $\mathbb{IR}^n$, and $\mathbb{IR}^{n\times n}$ the set of real closed intervals, n-dimensional interval vectors and $n\times n$ interval matrices, respectively. If $x=[\underline x,\bar x]\in\mathbb{IR}^n$ then we write $\operatorname{mid} x:=\check x:=(\bar x+\underline x)/2$ for the *midpoint*, $\operatorname{rad} x:=\rho(x):=(\bar x-\underline x)/2$ for the *radius*, $|x|:=\sup\{\bar x,-\underline x\}=\sup\{|\tilde x|\,|\,\tilde x\in x\}$ for the *absolute value*, and $\hat x:=x-\check x=[-\rho(x),\rho(x)]$ for the *radial part* of x; similar definitions apply to interval matrices. The *Ostrowski operator* $\langle\cdot\rangle$ associates with a square matrix $A\in\mathbb{IR}^{n\times n}$ the matrix $A'=\langle A\rangle$ with entries $A'_{ii}:=\min\{|\tilde a|\,|\,\tilde a\in A_{ii}\}$, $A'_{ik}:=-|A_{ik}|$ for $i\neq k$. In terms of the *Jacobi splitting* $A=D-E$ of A, defined by $D_{ii}:=A_{ii}$, $E_{ii}:=0$ and $D_{ik}:=0$, $E_{ik}:=-A_{ik}$ for $i\neq k$ we have

$$\langle A\rangle=\langle D\rangle-|E|,\quad |A|=|D|+|E|\geqq\langle D\rangle+|E|.$$

Sum and *difference* of two bounded subsets $\Sigma_1,\Sigma_2\subseteq\mathbb{R}^n$ are defined as $\Sigma_1\pm\Sigma_2:=\{\tilde x\pm\tilde y\,|\,\tilde x\in\Sigma_1,\ \tilde y\in\Sigma_2\}$, and the *hull* of a bounded subset $\Sigma\subseteq\mathbb{R}^n$ is defined as $\Box\Sigma:=[\inf\Sigma,\sup\Sigma]$, the smallest interval vector containing Σ. The (vector valued) *distance* of two interval vectors $x,y\in\mathbb{IR}^n$ is defined as

$$q(x,y):=\inf\{q\in\mathbb{R}^n\,|\,q\geqq 0,\ x\subseteq y+[-q,q],\ y\subseteq x+[-q,q]\}. \tag{2}$$

One immediately establishes the following properties of the distance.

LEMMA 1. *Let $x,y,z\in\mathbb{IR}^n$. Then*

$$q(x,y)=0\quad \textit{iff } x=y,$$
$$q(y,x)=q(x,y),$$
$$q(x,z)\leqq q(x,y)+q(y,z).$$

LEMMA 2. *Let $x,y\in\mathbb{IR}^n$. Then*

$$x\subseteq y\quad \textit{iff } |\check x-\check y|\leqq\rho(y)-\rho(x), \tag{3}$$
$$q(x,y)=|\check x-\check y|+|\rho(x)-\rho(y)|. \tag{4}$$

Proof. Equation (3) is well known, and the equivalence of (2) and (4) is proved in Lemma 1 of Neumaier [13]. □

Note that often (4) is taken as a definition of the distance of intervals.

LEMMA 3. *If $f(\xi_1,\cdots,\xi_m)$ is an arithmetic expression in m variables such that the variables $\xi_1,\cdots,\xi_n$ (where $n\leqq m$) occur only once then for $x_1,\cdots,x_m\in\mathbb{IR}$, $\rho(x_i)=0$ for $i>n$,*

$$f(x_1,\cdots,x_m)=\Box\{f(\xi_1,\cdots,\xi_m)\,|\,\xi_i\in x_i\ (i=1,\cdots,m)\}.$$

In particular, if $a,b\in\mathbb{IR}^n$ then

$$a+b=\{\tilde a+\tilde b\,|\,\tilde a\in a,\ \tilde b\in b\},$$

and if $A\in\mathbb{IR}^{n\times n}$, $\tilde x\in\mathbb{R}^n$ then

$$A\tilde x=\{\tilde A\tilde x\,|\,\tilde A\in A\}.$$

Proof. Well known; see [1] or [11]. □

We define a *skew vector* as a set of the form

$$\{\tilde x+C\tilde r\,|\,\tilde r\in r\}, \tag{5}$$

where $\tilde x\in\mathbb{R}^n$, $C\in\mathbb{R}^{n\times n}$ and $\tilde r\in\mathbb{IR}^n$. Geometrically, a skew vector is a (possibly degenerate) parallelepiped with axes parallel to the columns of C. Skew vectors occur in the literature on interval differential equations, hidden behind the concept of a moving coordinate frame (see e.g. Moore [11], Eijgenraam [4]). The hull of the skew vector (5) is easily seen to be the interval $\tilde x+Cr$. In case of large off-diagonal entries in C the distance between a skew vector and its hull may be quite large; cf. Fig. 1, which shows a skew vector (5) and its hull, where

$$\tilde x=\begin{pmatrix}2\\2\end{pmatrix},\quad C=\begin{pmatrix}1&0\\1&0\end{pmatrix},\quad r=\begin{pmatrix}[-1,1]\\ [-1,1]\end{pmatrix}.$$

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[1] To emphasize the nature of C as a *fixed* real matrix, we do not write $\check C$ which would be the "generic" notation.

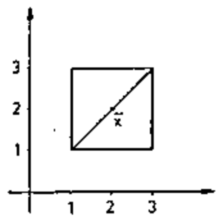

FIG. 1

Therefore it can be expected that, in general, the overestimation of a set Σ by an enclosing interval (e.g. its hull) can be considerably reduced by enclosing Σ in a suitable skew vector.

2. The solution set of linear interval equations. Let $A \in \mathbb{IR}^{n\times n}$, $b \in \mathbb{IR}^n$, and suppose that A is regular. We are interested in narrow enclosures of the set

$$\Sigma(A, b) := \{\tilde{A}^{-1}\tilde{b} \mid \tilde{A} \in A, \tilde{b} \in b\}$$

of solutions of the linear system of interval equations

$$\tilde{A}\tilde{x} = \tilde{b} \qquad (\tilde{A} \in A, \tilde{b} \in b).$$

The optimal interval enclosure is

$$A^H b := \square\Sigma(A, b) = \square\{\tilde{A}^{-1}\tilde{b} \mid \tilde{A} \in A, \tilde{b} \in b\};$$

but in practice often only cruder enclosures are available. In this section we consider a particular enclosure based on the residual $r := b - A\tilde{x}$ of an approximation $\tilde{x}$ to the solution.

THEOREM 1. *Suppose that $C \in \mathbb{R}^{n\times n}$ is such that CA is regular. Let $\tilde{x} \in \mathbb{R}^n$, $r := b - A\tilde{x}$, and suppose that $d \in \mathbb{IR}^n$ satisfies*

$$A^H((I - AC)r) \subseteq d \quad or \quad (CA)^H((I - CA)(Cr)) \subseteq d.$$

Then $x := \tilde{x} + Cr + d$ satisfies

(1) $$A^H b \subseteq x \subseteq A^H b + (d - d);$$

in particular,

(2) $$q(A^H b, x) \leqq 2\rho(d),$$

(3) $$0 \leqq \operatorname{rad} x - \operatorname{rad} A^H b \leqq 2\rho(d).$$

Proof. By assumption, C and A are regular. Let $\tilde{A} \in A$, $\tilde{b} \in b$ and put

(4) $$\tilde{r} := \tilde{b} - \tilde{A}\tilde{x}, \quad \tilde{y} := \tilde{x} + C\tilde{r}, \quad \tilde{z} := \tilde{A}^{-1}\tilde{b}.$$

Then $\tilde{z} - \tilde{y} = \tilde{A}^{-1}\tilde{b} - \tilde{x} - C(\tilde{b} - \tilde{A}\tilde{x}) = \tilde{A}^{-1}(I - \tilde{A}C)(\tilde{b} - \tilde{A}\tilde{x})$, hence

(5) $$\tilde{z} - \tilde{y} = \tilde{A}^{-1}(I - \tilde{A}C)\tilde{r} = (C\tilde{A})^{-1}(I - C\tilde{A})C\tilde{r} \in d.$$

Since $\tilde{y} \in \tilde{x} + Cr$, this implies (1). To prove (2) and (3), we use the fact that Lemma 3 implies

$$r = \{\tilde{b} - \tilde{A}\tilde{x} \mid \tilde{A} \in A, \tilde{b} \in b\};$$

hence every $\tilde{r} \in r$ and every $\tilde{y}$ in

(6) $$\tilde{\Sigma} := \{\tilde{x} + C\tilde{r} \mid \tilde{r} \in r\}$$

can be written in the form (4). Therefore $\Sigma(A, B) \subseteq \tilde{\Sigma} + d \subseteq \tilde{x} + Cr + d = x$ and $\tilde{\Sigma} \subseteq \Sigma(A, B) - d \subseteq A^H b - d$ whence $\tilde{x} + Cr \subseteq A^H b - d$ and $x = \tilde{x} + Cr + d \subseteq A^H b - d + d$. This implies (1), and (2), (3) are immediate consequences. □

For practical application, C should be an approximation of the inverse of some $\tilde{A} \in A$ so that $CA \approx I$, and $\tilde{x}$ should be an approximation of $\tilde{A}^{-1}\tilde{b}$ for some $\tilde{b} \in b$ so that $A\tilde{x} \approx b$. If the residuals $I - CA$ and $b - A\tilde{x}$ are of order $O(\varepsilon)$ then d can be taken of order $O(\varepsilon^2)$, so that (1) overestimates the hull by only $O(\varepsilon^2)$. Note that the amount of overestimation can be computed a posteriori from (3).

The proof of Theorem 1 suggests that $\Sigma(A, b)$ itself may be enclosed with little overestimation by a skew vector. This is indeed the case.

THEOREM 2. *Suppose that $C \in \mathbb{R}^{n\times n}$ is such that AC is regular. Let $\tilde{x} \in \mathbb{R}^n$, $r := b - A\tilde{x}$ and suppose that*

$$(AC)^H((I - AC)r) \subseteq e \in \mathbb{IR}^n.$$

Then $\Sigma_0 := \{\tilde{x} + C\tilde{p} \mid \tilde{p} \in r + e\}$ satisfies

(7) $$\Sigma(A, b) \subseteq \Sigma_0 \subseteq \Sigma(A, b) + |C|(e - e).$$

Proof. As before,

$$\tilde{e} := C^{-1}(\tilde{z} - \tilde{y}) = (\tilde{A}C)^{-1}(I - \tilde{A}C)\tilde{r} \in e,$$

so that $\tilde{z} = \tilde{y} + C\tilde{e} = \tilde{x} + C(\tilde{r} + \tilde{e})$, which implies $\Sigma(A, b) \subseteq \Sigma_0$. Again as before, $\tilde{\Sigma} \subseteq \Sigma(A, b) - \{C\tilde{e} \mid \tilde{e} \in e\}$ so that $\Sigma_0 \subseteq \tilde{\Sigma} + \{C\tilde{e}' \mid \tilde{e}' \in e\} \subseteq \Sigma(A, b) + \{C(\tilde{e}' - \tilde{e}) \mid \tilde{e}, \tilde{e}' \in e\} \subseteq \Sigma(A, b) + |C|(e - e)$. □

Again, if the residuals $I - AC$ and $b - A\tilde{x}$ are of order $O(\varepsilon)$ then e can be chosen of order $O(\varepsilon^2)$, so that (6) overestimates the solution set by only $O(\varepsilon^2)$. Thus (6) can be expected to be a very accurate enclosure for the solution set. Note that the amount of overestimation can be calculated a posteriori from (7).

3. The preconditioned fixpoint inverse. $A \in \mathbb{IR}^{n\times n}$ is called an *H-matrix* if $\langle A\rangle u > 0$ for some $u > 0$; cf. Neumaier [12, § 2.5]. In [12], the *fixpoint inverse* of an H-matrix $A \in \mathbb{IR}^{n\times n}$ was defined as the sublinear map $A^F : \mathbb{IR}^n \to \mathbb{IR}^n$ which maps $b \in \mathbb{IR}^n$ to the unique solution $x = A^F b$ of the fixpoint equation

$$x = L^F(b + Ex),$$

where $A = L - E$ is a direct splitting of A. By [12, Prop. 11], the vector $x = A^F b$ is an enclosure of the hull $A^H b$. Here we consider the preconditioned equations

$$C\tilde{A}\tilde{x} = C\tilde{b} \qquad (\tilde{A} \in A, \tilde{b} \in b),$$

and, assuming that CA is an H-matrix, the corresponding enclosure $x = (CA)^F(Cb)$ for $(CA)^H(Cb)$. The norm used in the following is the maximum norm.

THEOREM 3. *Suppose that $C \in \mathbb{R}^{n\times n}$ is such that CA is an H-matrix. Then*

(1) $$z := A^H b \subseteq (CA)^H(Cb) \subseteq (CA)^F(Cb) \subseteq \check{z} + \langle CA\rangle^{-1}|CA|\hat{z}.$$

Moreover, if CA is strictly diagonally dominant, i.e.,

$$\beta := \max_i \sum_{k \neq i} \frac{|CA|_{ik}}{\langle CA\rangle_{ii}} < 1$$

then

(2) $$\|\operatorname{rad} A^H b\| \leqq \|\operatorname{rad} (CA)^H(Cb)\| \leqq \|\operatorname{rad} (CA)^F(Cb)\| \leqq \frac{1+\beta}{1-\beta}\|\operatorname{rad} A^H b\|.$$

Proof. Let $CA = D - E$ be the Jacobi splitting of CA. Then $x := (CA)^F(Cb)$ satisfies the equation

$$x = D^F(Cb + Ex). \tag{3}$$

Fix $i \in \{1, \cdots, n\}$. The splitting equation implies

$$D_{ii} = \sum_j C_{ij}A_{ji}, \quad E_{ik} = -\sum_j C_{ij}A_{jk} \quad (k \neq i),$$

and (3) gives

$$x_i = \sum_k \frac{C_{ik}b_k + E_{ik}x_k}{D_{ii}}.$$

In these expressions, each variable apart from the $C_{ij} \in \mathbb{R}$ occurs only once (note that i is fixed); hence these expressions equal the range (Lemma 3), and for each $\xi \in x_i$ we can find $\tilde{x} \in x$, $\tilde{b} \in b$, $\tilde{D} \in D$, $\tilde{E} \in E$, $\tilde{A} \in A$ such that $C\tilde{A} = \tilde{D} - \tilde{E}$ and $\tilde{y} := \tilde{D}^{-1}(C\tilde{b} + \tilde{E}\tilde{x})$ satisfies $\tilde{y}_i = \xi$.

Now $\tilde{z} := \tilde{A}^{-1}\tilde{b} \in A^H b = z$ and $\tilde{y} = \tilde{D}^{-1}(C\tilde{A}\tilde{z} + \tilde{E}\tilde{x}) = \tilde{D}^{-1}((\tilde{D} - \tilde{E})\tilde{z} + \tilde{E}\tilde{x}) = \tilde{z} + \tilde{D}^{-1}\tilde{E}(\tilde{x} - \tilde{z}) \in z + D^F E(x - z)$; therefore $\xi \in (z + D^F E(x - z))_i$, and since $\xi \in x_i$ and i were arbitrary we find

$$x \subseteq z + D^F E(x - z).$$

This implies

$$|x - \check{z}| \leqq |\hat{z} + D^F E(x - z)| \leqq \rho(z) + \langle D\rangle^{-1}|E||x - z|,$$

$$\langle D\rangle|x - \check{z}| \leqq \langle D\rangle\rho(z) + |E|(|x - \check{z}| + \rho(z)).$$

Therefore

$$|x - \check{z}| \leqq (\langle D\rangle - |E|)^{-1}(\langle D\rangle + |E|)\rho(z). \tag{4}$$

In particular,

$$|x - \check{z}| \leqq \langle CA\rangle^{-1}|CA|\rho(z), \qquad x \in \check{z} + \langle CA\rangle^{-1}|CA|\hat{z},$$

and (1) follows. Finally if CA is strictly diagonally dominant then $B := \langle D\rangle^{-1}|E|$ satisfies $\|B\| \leqq \beta < 1$ whence

$$\|\text{rad } x\| \leqq \|x - \check{z}\| \leqq \|(\langle D\rangle - |E|)^{-1}(\langle D\rangle + |E|)\rho(z)\|$$

$$= \|(I - B)^{-1}(I + B)\rho(z)\| \leqq \frac{1+\beta}{1-\beta}\|\rho(z)\|$$

which implies (2). □

Remarks. 1) If $\rho(A) = O(\varepsilon)$ and $C = \check{A}^{-1}$ then $CA = I + \check{A}^{-1}\hat{A} = I + O(\varepsilon)$ whence $\beta = O(\varepsilon)$. From (2) we therefore get

$$\frac{\text{rad }((CA)^F(Cb))}{\text{rad } A^H b} = 1 + O(\varepsilon),$$

i.e., very little overestimation.

2) For a computable bound on the overestimation we simply replace $A^H b$ in (2) by the slightly bigger $(CA)^F(Cb)$.

3) Theorem 3 implies that preconditioning with a good approximation C of $\check{A}^{-1}$ (resulting in a small β, say $\beta \leqq \frac{1}{2}$) increases the radius of $A^H b$ only by a small factor $\leqq (1+\beta)/(1-\beta) \leqq 3$, thus confirming the well-known observation that preconditioning usually gives excellent enclosures. An asymptotic result of this type is already known for some time: Miller [10] proved

$$\text{rad }((CA)^H(Cb)) = \text{rad } A^H b + O(\varepsilon^2)$$

if $C = \check{A}^{-1}$ and $\rho(A)$, $\rho(b) = O(\varepsilon)$; cf. the exposition in Alefeld and Herzberger [1, Chap. 16].

4) Since $q(x, z) = |\check{x} - \check{z}| + |\rho(x) - \rho(z)| = |x - \check{z}| - \rho(z)$ (note $\rho(x) \geqq \rho(z)$ since $z \subseteq x$), we get from (4) the relation

$$q(x, z) \leqq 2(\langle D\rangle - |E|)^{-1}|E|\rho(z),$$

which in the strictly diagonally dominant case implies

$$\|q((CA)^F(Cb), A^H b)\| \leqq \frac{2\beta}{1-\beta}\|\text{rad } A^H b\|.$$

5) If $\|I - CA\| \leqq \beta_0 < 1$ then the iteration $x^{l+1} = Cb + Ex^l$, where $E := I - CA$, converges to a limit x^* satisfying $x^* = Cb + Ex^*$ and

$$z \subseteq (CA)^F(Cb) \subseteq x^*;$$

see Mayer [9], Gay [5] and Neumaier [12]. Proceeding as in the proof of Theorem 3 but with the splitting $CA = I - E$ one obtains the relation

$$z \subseteq x^* \subseteq \check{z} + (I - |E|)^{-1}(I + |E|)\hat{z},$$

and as in Remark 4 one gets

$$\|q(x^*, A^H b)\| \leqq \frac{2\beta_0}{1-\beta_0}\|\text{rad } A^H b\|. \tag{5}$$

6) Relation (5) improves Theorem 4.1 of Gay [5], who states almost the same inequality for $\beta_0 < \frac{1}{3}$, but with the denominator $1 - 3\beta_0$. However his proof seems to have a gap: In [5, p. 868, line 4] one can conclude $w(x^1) = w(d) + w(\square(Ed))$ but it is not clear whether this implies the required relation $w(x^1) = w(d) + \sup(\|\square(Ed)\|)$.

7) It would be interesting to have a result similar to Theorem 3 for preconditioned Gauss elimination.

4. On the choice of the preconditioning matrix. In Neumaier [12], the optimal choice of the preconditioning matrix C was discussed. It was shown that if $\|I - CA\| < 1$ for some $C \in \mathbb{R}^{n\times n}$ then $\|I - CA\|$ takes its minimum for $C = \check{A}^{-1}$. Here we present another optimality criterion based on an upper bound for the preconditioned fixpoint inverse or Gauss inverse. We remind the reader that $A^G b$ denotes the result of interval Gauss elimination applied without pivoting to the matrix A and the right-hand side b; cf. [12, Chap. 5].

THEOREM 4. *Let $A \in \mathbb{IR}^{n\times n}$, $b \in \mathbb{IR}^n$, and $C \in \mathbb{R}^{n\times n}$, and suppose that CA is an H-matrix. Then*

$$|(CA)^F(Cb)| \leqq \langle CA\rangle^{-1}|Cb|, \tag{1}$$

$$|(CA)^G(Cb)| \leqq \langle CA\rangle^{-1}|Cb|, \tag{2}$$

and if $\check{b} = 0$ then equality holds in (1). Moreover, $\check{A}^{-1}A$ is an H-matrix and

$$\langle \check{A}^{-1}A\rangle^{-1}|\check{A}^{-1}b| \leqq \langle CA\rangle^{-1}|Cb|. \tag{3}$$

Proof. By Theorem 9 of [12], $|(CA)^F| = (CA)^{-1}$ so that $|(CA)^F(Cb)| \leq |(CA)^F||Cb| = (CA)^{-1}|Cb|$, and (1) holds. If $\check{b} = 0$ then $a := Cb$ satisfies $\check{a} = 0$, and since $(CA)^F$ is normal [12, Chap. 4], (R8) in [12] implies $(CA)^F(Cb) = |(CA)^F|(Cb) = \langle CA\rangle^{-1}(Cb)$, so that (1) holds with equality. If, instead, Theorems 3 and 4 of [12] are used one obtains in the same way relation (2).

We now observe that CA is an H-matrix, so that by [12, Thm. 5], $\check{A}^{-1}A$ is an H-matrix. To prove (3), let $u > 0$ be such that $\langle CA\rangle u > 0$. Then $0 \notin (CA)_{ii}$ for $i = 1, \cdots, n$ so that by [12], § 2.2 (O4) and § 1.3, (B9) and (B14) (the last rule contains a misprint and should read $\rho(\mathrm{BA}) = \rho(\mathrm{B})|\mathrm{A}|$), we have

$$\langle CA\rangle = \langle C\check{A}\rangle - |C|\rho(A).$$

Therefore the matrix

$$P := \langle C\check{A}\rangle^{-1}\langle CA\rangle = I - \langle C\check{A}\rangle^{-1}|C|\rho(A)$$

satisfies $P \leqq I$, $Pu = \langle C\check{A}\rangle^{-1}\langle CA\rangle u > 0$. Hence P is an M-matrix; in particular, $P^{-1} \geqq 0$. Since

$$\langle \check{A}^{-1}A\rangle = \langle I + \check{A}^{-1}\mathring{A}\rangle \geqq I - |\check{A}^{-1}|\rho(A)$$

$$= I - |(C\check{A})^{-1}C|\rho(A) \geqq I - \langle C\check{A}\rangle^{-1}|C|\rho(A) = P,$$

we get $P\langle \check{A}^{-1}A\rangle^{-1} \leqq I$; hence $\langle \check{A}^{-1}A\rangle^{-1} \leqq P^{-1}$, and finally

$$\langle \check{A}^{-1}A\rangle^{-1}|\check{A}^{-1}b| \leqq P^{-1}|\check{A}^{-1}b| = P^{-1}|(C\check{A})^{-1}Cb|$$

$$\leqq P^{-1}\langle C\check{A}\rangle^{-1}|Cb| = \langle CA\rangle^{-1}|Cb|. \quad \square$$

Remarks. 1) Inequality (3) describes a minimality property of the upper bound $\langle CA\rangle^{-1}|Cb|$. It would be interesting to know an optimal C for which $|(CA)^F(Cb)|$ and $|(CA)^G(Cb)|$ themselves take their minimum. It seems that at least for H-matrices A and $(CA)^G(Cb)$, the optimal choice is not $C = \check{A}^{-1}$ but $C = \tilde{A}^{-1}$ where $\tilde{A} \in A$ satisfies

$$|\tilde{A}_{ii}| = \max(|\underline{A}_{ii}|, |\bar{A}_{ii}|),$$

$$|\tilde{A}_{ik}| = \min(|\underline{A}_{ik}|, |\bar{A}_{ik}|) \quad \text{for } i \neq k.$$

2) A result similar to Theorem 4 is implicitly in Krawczyk [8]: If the spectral radius of $|I - CA|$ is less than one then the expression $(I - |I - CA|)^{-1}|Cb|$ (which is also an upper bound for $(CA)^F(Cb)$) takes its minimum for $C = \check{A}^{-1}$; indeed this follows by multiplying the third inequality in the lemma in [8] with $(e - \bar{r})^{-1}$ which in the present notation is $(I - |I - \check{A}^{-1}A|)^{-1}$.

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