On the Comparison of *H*-matrices with *M*-matrices

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ABSTRACT

An old theorem of Ostrowski states that the absolute value of the inverse of an H-matrix is, componentwise, bounded by the inverse of a related M-matrix. In this note we discuss the quality of this bound and a related bound for triangular decompositions.

In his famous paper, Ostrowski [12] associates with every matrix $A=(a_{ik})$ the *comparison matrix*

$$\langle A\rangle := (\varepsilon_{ik}|a_{ik}|), \qquad \varepsilon_{ik}=\begin{cases} 1 & \text{if} \quad i=k, \\ -1 & \text{otherwise}. \end{cases} \tag{1}$$

$\langle A\rangle$ is obtained from A if we replace the diagonal entries by their absolute values, and the off-diagonal entries by their negative absolute values. The matrix A is called an *M-matrix* if $\langle A\rangle = A$ and if there is a vector $u>0$ with $Au>0$ (inequalities and, later, absolute values are understood componentwise); many other equivalent definitions have been given in the literature (see e.g. Berman and Plemmons [2]). Ostrowski [12] showed that every M-matrix A is nonsingular and inverse positive, i.e. $A^{-1}\geqslant 0$. The matrix A is called an *H-matrix* if there is a vector $u>0$ with $\langle A\rangle u>0$. Ostrowski [12] also showed that for H-matrices

$$|A^{-1}|\leqslant\langle A\rangle^{-1}. \tag{2}$$

Equation (2) is relevant in the context of estimating matrix condition

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numbers and constructing error bounds for solutions of equations; see e.g. Anderson and Karasalo [1], Higham [6, 7], Karasalo [8], Manteuffel [9], and Neumaier [10, 11]. The purpose of this note is to discuss the amount of overestimation in (2) and a related sharper bound involving the triangular decomposition of A.

THEOREM 1. *Let A be an H-matrix and*

$$\Delta := I - \operatorname{Diag}(A)^{-1}A, \tag{3}$$

$$\Omega := (I - |\Delta|)^{-1}(|\Delta| - \Delta). \tag{4}$$

Then $\Omega \geqslant 0$ and

$$|A^{-1}| \leqslant \langle A \rangle^{-1} \leqslant (I + \Omega)|A^{-1}|. \tag{5}$$

Proof. Let $D := \operatorname{Diag}(A)$ be the diagonal matrix whose diagonal entries agree with that of A. Then

$$A = D(I - \Delta), \qquad \langle A \rangle = |D|(I - |\Delta|).$$

If we put $B := I - \Delta$, then $\langle B \rangle = I - |\Delta|$ and

$$I + \Omega = (I - |\Delta|)^{-1}(I - |\Delta| + |\Delta| - \Delta) = \langle B \rangle^{-1}B.$$

Hence $\langle A \rangle^{-1} = \langle B \rangle^{-1}|D|^{-1} = (I + \Omega)B^{-1}|D|^{-1} = (I + \Omega)A^{-1}D|D|^{-1}$ and $\langle A \rangle^{-1} \leqslant (I + \Omega)|A^{-1}||D||D|^{-1} = (I + \Omega)|A^{-1}|$, giving the upper bound in (5). The lower bound is just (2). ∎

For applications to error estimation (Neumaier [10]), bounds for $\||A^{-1}|c\|$ are required; e.g. we have $\|A^{-1}\|_\infty = \||A^{-1}|e\|_\infty$, where $e = (1, \dots, 1)^T$.

COROLLARY. *Under the hypothesis of Theorem 1,*

$$\||A^{-1}|c\| \leqslant \|\langle A \rangle^{-1}c\| \leqslant (1 + \omega)\||A^{-1}|c\| \tag{6}$$

for all $c \geqslant 0$ and every monotone matrix norm with $\|I\| = 1$ and $\|\Delta\| < 1$;

here

$$\omega = \frac{\| |\Delta| - \Delta \|}{1 - \||\Delta|\|}. \tag{7}$$

Proof. $\|\Omega\| \leqslant \omega.$ ∎

REMARKS.

1. Since A is an H-matrix, there is $u > 0$ such that $\langle A \rangle u > 0$; hence $|\Delta| u < u$ and the scaled maximum norm

$$\|B\|_u := \max_i \frac{(|B|u)_i}{u_i}$$

satisfies $\|I\|_u = 1$ and $\|\Delta\|_u < 1$.

2. Suppose that the diagonal elements of A are positive. Then $\Omega = 0$ iff A is an M-matrix; hence Ω and ω can be interpreted as a measure for the "departure of A from an M-matrix."

The bound (2) can be improved by using triangular decompositions of A. The restriction that A is an H-matrix can then be dropped.

THEOREM 2. *Let A be a nonsingular $n \times n$ matrix possessing a decomposition $A = LR$ into the product of two triangular matrices. Then*

$$|A^{-1}| \leqslant \langle R \rangle^{-1} \langle L \rangle^{-1}. \tag{8}$$

Moreover, if A is an H-matrix then

$$|A^{-1}| \leqslant \langle R \rangle^{-1} \langle L \rangle^{-1} \leqslant \langle A \rangle^{-1}. \tag{9}$$

The *proof* is based on the following

LEMMA. *If $A = (a_{ik})$, $B = (b_{ik})$, and for $i, k = 1, \ldots, n$,*

$$a_{ii} a_{ik} b_{ik} b_{kk} \geqslant 0, \tag{10}$$

then

$$\langle AB \rangle \leqslant \langle A \rangle \langle B \rangle. \tag{11}$$

Proof. We compare the (i,k) entries of $\langle AB\rangle$ and $\langle A\rangle\langle B\rangle$. For $i=k$ we have $[\langle AB\rangle]_{ii}=|\Sigma_j a_{ij}b_{ji}|\leqslant\Sigma_j|a_{ij}||b_{ji}|=[\langle A\rangle\langle B\rangle]_{ii}$.

For $i\neq k$ we put

$$p:=\sum_{j\neq i,k}a_{ij}b_{jk},\qquad q:=a_{ii}b_{ik},\qquad r:=a_{ik}b_{kk}.$$

By hypothesis, q and r have the same sign. Therefore $|q|+|r|=|q+r|=|p+q+r-p|\leqslant|p+q+r|+|p|=-[\langle AB\rangle]_{ik}+|p|$, whence $[\langle AB\rangle]_{ik}\leqslant|p|-|q|-|r|\leqslant\Sigma_{j\neq i,k}|a_{ij}||b_{jk}|-|a_{ii}||b_{ik}|-|a_{ik}||b_{kk}|=[\langle A\rangle\langle B\rangle]_{ik}$. ■

Proof of Theorem 2. Since A is nonsingular, L and R are nonsingular. But by Schröder [13, p. 42, Example 2.1], every nonsingular triangular matrix is an H-matrix; therefore $|A^{-1}|=|R^{-1}L^{-1}|\leqslant|R^{-1}||L^{-1}|\leqslant\langle R\rangle^{-1}\langle L\rangle^{-1}$, and (8) follows. Now the lemma, applied with L and R in place of A and B shows that

$$\langle A\rangle=\langle LR\rangle\leqslant\langle L\rangle\langle R\rangle.$$

If now A is an H-matrix, then $\langle A\rangle^{-1}\geqslant 0$; hence $I\leqslant\langle A\rangle^{-1}\langle L\rangle\langle R\rangle$, and since $\langle R\rangle^{-1},\langle L\rangle^{-1}\geqslant 0$, this implies $\langle R\rangle^{-1}\langle L\rangle^{-1}\leqslant\langle A\rangle^{-1}$ giving (9). ■

In certain cases, equality holds in (8) and (9). If A is an M-matrix, then $\langle A\rangle^{-1}=A^{-1}=|A^{-1}|$, so that (9) also implies $|A^{-1}|=\langle R\rangle^{-1}\langle L\rangle^{-1}$. This also follows from the fact (Fiedler and Pták [3]) that the triangular factors of an M-matrix are themselves M-matrices, whence $|A^{-1}|=A^{-1}=R^{-1}L^{-1}=\langle R\rangle^{-1}\langle L\rangle^{-1}$. More generally, equality holds in (8) and (9) whenever A is sign equivalent to an M-matrix, i.e. if $A=\Sigma_1 B\Sigma_2$ with an M-matrix B and diagonal matrices Σ_1,Σ_2 with $|\Sigma_1|=|\Sigma_2|=I$; indeed, all terms of (9) are invariant under such sign changes. In particular, equality holds in (8) and (9) for symmetric, positive definite tridiagonal matrices A, since these are sign equivalent to M-matrices; cf. Higham [6].

Certain nonsymmetric tridiagonal matrices which are diagonally equivalent to symmetric positive definite matrices also satisfy $|A^{-1}|=\langle R\rangle^{-1}\langle L\rangle^{-1}$, but for the "tridiagonal" 2×2 matrix

$$A=\begin{pmatrix}1&1\\-1&1\end{pmatrix}$$

we have

$$A^{-1}=\tfrac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix},$$

$$\langle R\rangle^{-1}\langle L\rangle^{-1}=\tfrac{1}{2}\begin{pmatrix}3&1\\1&1\end{pmatrix}\neq|A^{-1}|.$$

It is not difficult to show that for a 2×2 matrix

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad |ad|\geqslant |bc| \tag{12}$$

we always have

$$\langle R\rangle^{-1}\langle L\rangle^{-1}\leqslant 3|A^{-1}|,$$

and by a permutation of rows, an arbitrary 2×2 matrix can be brought into the form (12). But the example of the symmetric, balanced matrix

$$A=\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A^{-1}=\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \qquad \langle R\rangle^{-1}\langle L\rangle^{-1}=\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

shows that the classical pivoting strategies do not produce the optimum; an interchange of the two rows leads to

$$A=\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

$$A^{-1}=\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad \langle R\rangle^{-1}\langle L\rangle^{-1}=\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}=|A^{-1}|.$$

For matrices of order $n>2$ it is not clear how an optimal pivoting strategy should look.

For strictly diagonally dominant matrices an easily computable upper bound for the overestimation in (8) can be obtained from (6) and (9). We relate to $A=(a_{ik})$ the numbers

$$\beta:=\max_{i=1,\dots,n}|a_{ii}|^{-1}\sum_{k\neq i}|a_{ik}|, \tag{13}$$

$$\alpha:=\max_{i=1,\dots,n}|a_{ii}|^{-1}\sum_{\substack{k\neq i\\ a_{ii}a_{ik}>0}}|a_{ik}|. \tag{14}$$

Then $\alpha\leqslant\beta$, and in the notation of Proposition 3.1 we have $\|\Delta\|_\infty=\beta$, $\||\Delta|-\Delta\|_\infty=2\alpha$; if A is strictly diagonally dominant, then $\beta<1$, whence by

(9) and (6),

$$\| |A^{-1}|c \|_\infty \leqslant \| \langle R \rangle^{-1} \langle L \rangle^{-1} c \|_\infty \leqslant \left(1 + \frac{2\alpha}{1-\beta}\right) \| |A^{-1}|c \|_\infty. \tag{15}$$

If α is small and β is not too close to 1, then (15) shows that $\langle A \rangle^{-1}$ is a reasonable bound for $|A^{-1}|$. Note that for M-matrices, $\alpha = 0$.

REMARK. If A is not diagonally dominant, then (8) can grossly overestimate $|A^{-1}|$, an extreme case being certain nonnegative matrices A. Exponential overestimation is possible even for well-conditioned matrices. An example for this is the Frank matrix [4; 5, Example 3.12]

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & n \end{pmatrix}$$

with

$$L = R^T = \begin{pmatrix} 1 & & & \\ 1 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ 0 & & & & -1 & 1 \end{pmatrix},$$

where

$$\langle L \rangle^{-1} = \left(\langle R \rangle^{-1}\right)^T = \begin{pmatrix} 1 & & & & & & 0 \\ 1 & 1 & & & & & \\ 2 & 1 & \ddots & & & & \\ 4 & 2 & \ddots & \ddots & & & \\ \vdots & \vdots & & \ddots & \ddots & & \\ \vdots & \vdots & & & \ddots & \ddots & \\ 2^{n-2} & 2^{n-3} & \cdots & \cdots & 2 & 1 & 1 \end{pmatrix},$$

whence $\|A^{-1}\|_\infty = 4$, $\|\langle R\rangle^{-1}\langle L\rangle^{-1}\|_\infty = 1+2+2^3+2^5+\cdots+2^{2n-3} = (4^n+2)/6$.

On the other hand, if the matrix A is nonnegative but diagonally dominant, then $\alpha=\beta$ in (13)–(15), and a severe overestimation is possible only if the factor

$$1+\frac{2\alpha}{1-\beta}=\frac{1+\beta}{1-\beta}$$

is large, i.e. if β is very close to 1. This theoretical possibility is however not observed in practice. Among 120 randomly generated 20×20 diagonally dominant matrices A with $\beta = 0.99$, 0.9999, and 0.999999 and random $c>0$, no case was found in which $\|\langle R\rangle^{-1}\langle L\rangle^{-1}c\|_\infty > 2\||A^{-1}|c\|_\infty$. It would be interesting to have an improved estimate of the type (15) explaining this.

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