Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 118, No. 1, August 15, 1986
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An Improved Interval Newton Operator

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In this paper interval operators producing a sequence of intervals are discussed. Each interval of such a sequence encloses a set of solutions of nonlinear systems of equations. For these interval operators we formulate theorems about existence and uniqueness of fixintervals. © 1986 Academic Press, Inc.

In dieser Arbeit werden Intervall-Operatoren diskutiert, die eine Folge von Intervallen erzeugen. Jedes Intervall einer solchen Folge schließt eine Lösungsmenge nichtlinearer Gleichungssysteme ein. Für diese Intervall-Operatoren formulieren wir Sätze über die Existenz und Eindeutigkeit von Fixintervallen. © 1986 Academic Press, Inc.

1. INTRODUCTION

Recently, a number of authors (Adams [1], Gay [5], Krawczyk [8]) considered the extension of various forms of the interval Newton method to sets of nonlinear systems of equations. Following [8] we consider here function strips, i.e., interval valued mappings $G(x)$ defined for vectors in a subset D of $\mathbb{R}^n$. A vector $x^* \in D$ is considered as a zero of G if $0 \in G(x^*)$. We study two related interval Newton operators $N(X)$ and $NV(X)$ with the property that every zero $x^* \in X \subseteq D$ of a Lipschitz continuous function strip G satisfies $x^* \in NV(X) \subseteq N(X)$. Interval operators with this property are commonly used to enclose iteratively the set of zeros of G in D by an iteration of the type (a) $X_{k+1} := N(X_k)$ or (b) $X_{k+1} := N(X_k) \cap X_k$. If the iteration (a) converges, its limit X^* is a fixinterval of the Newton operator; hence it is useful to have results about existence and uniqueness of such fixintervals.

0022-247X/86 $3.00

After some preparations in Section 2, whose main result (Proposition 2.1) is a simplified version of Rall's [11] midpoint-radius formulae for the product of two intervals, the interval Newton operators $N(X)$ and $NV(X)$ are introduced in Section 4 by means of

$$N(X) := \check{x} - aG(\check{x}) + [-q, q](aG(\check{x})),$$

$$NV(X) := \check{x} - [e-q, e+q](aG(\check{x})),$$

where e is the identity matrix and a and q are matrices determined by the Lipschitz condition. The zero-enclosing property is shown in Theorem 4.1. In the special case that the Lipschitz matrix is inverse isotone, the operator $NV(X)$ is compared with an optimal inclusion operator in Section 5.

Finally, Section 6 is concerned with existence and uniqueness results for fixintervals. Since the fixintervals of $N(X)$ and $NV(X)$ agree (Proposition 6.1), only $N(X)$ is discussed further. Uniqueness can be guaranteed only for the midpoint of the fixinterval (Proposition 6.2 and Example 6.1), or under the assumption of a constant Lipschitz matrix (Theorem 6.3). Moreover it is shown that if the radius of the domain D of G is sufficiently large then a fixinterval exists (Theorem 6.6 and Corollary).

2. Notation and Basic Concepts

Small letters denote real values, vectors, matrices, and real-valued vector functions (except: i, j, k, m, n for index notation). Capital letters denote intervals, interval vectors, interval matrices, and interval functions.

$\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^{m\times n}$ the set of all $m \times n$-matrices with 0 as zero in $\mathbb{R}^{m\times n}$. We consider the n-dimensional vector space $\mathbb{R}^n := \mathbb{R}^{n\times 1}$ and the set of real numbers $\mathbb{R} := \mathbb{R}^1$ as special cases. Let the "$\leqslant$"-relation be defined componentwise and let $\mathbb{R}_+^{m\times n}$ denote the positive cone with regard to the $\leqslant$-relation. The sign of a vector $x \in \mathbb{R}^n$ is defined as the vector sgn x whose nth component is 1 if $x_i > 0$, 0 if $x_i = 0$, and -1 if $x_i < 0$. If $a \in \mathbb{R}^{n\times n}$ then $\sigma(a)$ denotes the spectral radius of the matrix a.

Let an *interval* $A := [\underline{a}\, \bar{a}]$ with $\underline{a}\, \bar{a} \in \mathbb{R}^{m\times n}$ and $\underline{a} \leqslant \bar{a}$ be defined in the usual manner, i.e., $A := \{a \mid \underline{a} \leqslant a \leqslant \bar{a}\}$. $\mathbb{IR}^{m\times n}$ denotes the set of all intervals of $\mathbb{R}^{m\times n}$. If $A \in \mathbb{IR}^{m\times n}$ and $m > 1$, $n > 1$, then we also call A an *interval matrix*; we also call X an *interval vector* if $X \in \mathbb{IR}^n$ and $n > 1$. We call an interval matrix $A \in \mathbb{IR}^{n\times n}$ *regular*, if all $a \in A$ are regular. If $D \subseteq \mathbb{R}^n$ then $\mathbb{I}D := \{X \in \mathbb{IR}^n \mid X \subseteq D\}$. We identity a real value $a \in \mathbb{R}^{m\times n}$ and the degenerate interval $A = [a, a] \in \mathbb{IR}^{m\times n}$. Concerning interval arithmetic operations we refer to Alefeld and Herzberger [2].

Let $A \in \mathbb{IR}^{m \times n}$. Then

$$\operatorname{mid} A := \tfrac{1}{2}(\underline{a} - \bar{a}) \in \mathbb{R}^{m \times n} \tag{2.1}$$

defines the *midpoint of* A,

$$\operatorname{rad} A := \tfrac{1}{2}(\bar{a} - \underline{a}) \in \mathbb{R}_+^{m \times n} \tag{2.2}$$

defines the *radius of* A, and

$$|A| := \sup(\bar{a}, -\underline{a}) \in \mathbb{R}_+^{m \times n} \tag{2.3}$$

defines the *absolute value of* A.

Remark. Instead of mid A we also write $\check{a}$.

We shall make use of the following rules about the computation of midpoint, radius, and absolute value.

Let $X, Y \in \mathbb{IR}^n$, $A \in \mathbb{IR}^{n \times n}$, $a \in \mathbb{R}_+^{n \times n}$. Then

$$\operatorname{mid}(X \pm Y) = \operatorname{mid} X \pm \operatorname{mid} Y, \tag{2.4}$$

$$\operatorname{mid}[-a, a] = 0, \tag{2.5}$$

$$\operatorname{mid} A = 0 \Rightarrow \operatorname{mid}(AX) = 0, \tag{2.6}$$

$$\operatorname{rad} X = 0 \Leftrightarrow X = [x, x], \tag{2.7}$$

$$\operatorname{rad}(X \pm Y) = \operatorname{rad} X + \operatorname{rad} Y, \tag{2.8}$$

$$\operatorname{rad}[-a, a] = |[-a, a]| = a, \tag{2.9}$$

$$\operatorname{mid} A = 0 \Rightarrow \operatorname{rad}(AX) = \operatorname{rad} A \cdot |X|, \tag{2.10}$$

$$|X| = \operatorname{rad} X + |\operatorname{mid} X|, \tag{2.11}$$

$$X \subseteq Y \Leftrightarrow |\operatorname{mid} Y - \operatorname{mid} X| \leqslant \operatorname{rad} Y - \operatorname{rad} X. \tag{2.12}$$

Furthermore we need the following propositions.

PROPOSITION 2.1. *If $A \times \mathbb{IR}^{n \times n}$ is a diagonal matrix and $X \in \mathbb{IR}^n$ then*

$$\begin{aligned}\operatorname{mid}(AX) = \check{a}\check{x} + \operatorname{sgn}(\check{a}\check{x}) \inf\{&|\check{a}| \operatorname{rad} X, (\operatorname{rad} A)|\check{x}|,\\ &(\operatorname{rad} A)(\operatorname{rad} X)\},\end{aligned} \tag{2.13}$$

$$\begin{aligned}\operatorname{rad}(AX) = \sup\{&|\check{a}| \operatorname{rad} X + \operatorname{rad} A\ |\check{x}|,\\ &|\check{a}| \operatorname{rad} X + (\operatorname{rad} A)(\operatorname{rad} X),\\ &(\operatorname{rad} A)|\check{x}| + (\operatorname{rad} A)(\operatorname{rad} X)\}.\end{aligned} \tag{2.14}$$

Proof. Equations (2.13) and (2.14) are consequences from (3.8) in Rall [11], applied to each component of AX. ∎

PROPOSITION 2.2. *If* $A \in \mathbb{IR}^{n\times n}$ *is a diagonal matrix* $X \in \mathbb{IR}^n$ *then*

$$\operatorname{mid}(AX) = 0 \Leftrightarrow \operatorname{mid} A \cdot \operatorname{mid} X = 0. \tag{2.15}$$

Proof. From (2.13) follows $\operatorname{sgn}(\operatorname{mid}(AX)) = \operatorname{sgn}(\operatorname{mid} A \cdot \operatorname{mid} X)$, because the two terms of (2.13) have the same sign; so (2.15) is true. ∎

3. THE FUNCTION STRIP G AND ASSOCIATED LIPSCHITZ OPERATORS

Let $G \mid D \in \mathbb{IR}^n \to \mathbb{IR}^n$ be a map which associates with each $x \in D$ an interval

$$G(x) := [\underline{g}(x), \bar{g}(x)]. \tag{3.1}$$

We call such a map $G(x)$ a *function strip*.

We assume that both real functions $\underline{g}(x)$, $\bar{g}(x)$ satisfy a common interval Lipschitz condition

$$\underline{g}(x_1) - \underline{g}(x_2) \in L(X)(x_1 - x_2), \qquad \bar{g}(x_1) - \bar{g}(x_2) \in L(X)(x_1 - x_2)$$
$$\text{for all } x_1, x_2 \in X \in \mathbb{I}D, \tag{3.2}$$

where the Lipschitz matrix $L(X)$ is regular for all $X \in \mathbb{I}D$ and the operator L is continuous and isotone, i.e., $X \subseteq Y$ implies $L(X) \subseteq L(Y)$.

Furthermore we define

$$a(X) := (\operatorname{mid} L(X))^{-1} \tag{3.3}$$

and

$$r(X) := |a(X)| \operatorname{rad} L(X), \tag{3.4}$$

and we assume that

$$\sigma(r(X)) < 1 \qquad \text{for all} \quad X \in \mathbb{I}D. \tag{3.5}$$

Remark. It is sufficient to assume that $\operatorname{mid} L(D)$ is regular and $\sigma(r(D)) < 1$. For by Theorem 4 in [9] and Proposition 10 in [10] these simpler assumptions imply the regularity of $L(X)$ and $\sigma(r(X)) < 1$ for all $X \in \mathbb{I}D$.

In view of (3.5),

$$q(X) := r(X)(e - r(X))^{-1} \tag{3.6}$$

exists and $q(X) \geqslant 0$.

The nonnegative matrix $r(X)$ defined by (3.4) can be interpreted as a pseudometrical Lipschitz operator, because from the Lipschitz condition (3.2) follow Lipschitz conditions for the functions $x - \operatorname{mid}(a(X)\,G(x))$ and $\operatorname{rad}(a(X)\,G(x))$. They read

$$|(x_1 - \operatorname{mid}(a(X)\,G(x_1))) - (x_2 - \operatorname{mid}(a(X)\,G(x_2)))| \leqslant r(X)\,|x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in X \tag{3.7}$$

and

$$|\operatorname{rad}(a(X)\,G(x_1)) - \operatorname{rad}(a(X)\,G(x_2))| \leqslant r(X)\,|x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in X. \tag{3.8}$$

The proof of both Lipschitz conditions (3.7) and (3.8) is analogous to (3.16) in [9] and can be omitted here.

Later we need the following

LEMMA 3.1. *If $\sigma(r(Y)) < 1$ and $X \subseteq Y$ then the inequality*

$$(e - r(X))^{-1}\,|a(X)| \leqslant (e - r(Y))^{-1}\,|a(Y)| \tag{3.9}$$

holds.

Proof. With the abbreviations $a_x := a(X)$, $a_y = a(Y)$, $r_x := r(X)$, $r_y := r(Y)$, $\check{l}_x = \operatorname{mid} L(X)$, $\check{l}_y = \operatorname{mid} L(Y)$, and $b := a_y(\check{l}_x - \check{l}_y)$ (cf. (5.15)–(5.17) in [7]) it follows that $|a_x| \leqslant (e - |b|)^{-1}\,|a_y|$. Since $|b| \leqslant (r_y - r_x)(e - r_x)^{-1}$ and $(e - r_x)^{-1} \geqslant 0$ we obtain the inequality

$$\begin{aligned}(e - r_x)^{-1}\,|a_x| &\leqslant (e - r_x)^{-1}\,(e - (r_y - r_x)(e - r_x)^{-1})^{-1}\,|a_y| \\ &= ((e - (r_y - r_x)(e - r_x)^{-1})(e - r_x))^{-1}\,|a_y| \\ &= (e - r_y)^{-1}\,|a_y|. \quad \blacksquare\end{aligned}$$

4. INTERVAL NEWTON OPERATORS FOR THE FUNCTION STRIP G

Now we define an interval Newton operator

$$N(X) := \check{x} - a(X)\,G(\check{x}) + [-q(X), q(X)](a(X)\,G(\check{x})), \tag{4.1}$$

where G is a function strip with the domain D (see (3.1)), which satisfies an interval Lipschitz condition (3.2), $a(X)$ and $q(X)$ are defined by (3.3) and (3.6), with $r(X)$ defined by (3.4) satisfying the condition (3.5).

$N(X)$ is a continuous function, because $L(X)$, $a(X)$, $r(X)$, $q(X)$, and $G(X)$ are continuous functions.

From (4.1) follow by (2.4)–(2.10) the relations

$$\operatorname{mid} N(X) = \check{x} - \operatorname{mid}(a(X)\, G(\check{x})) \tag{4.2}$$

and

$$\operatorname{rad} N(X) = \operatorname{rad}(a(X)\, G(\check{x})) + q(X)\,|a(X)\, G(\check{x})|. \tag{4.3}$$

Remark. For the case that G degenerates to a function g the operator (4.1) was introduced in [6]. It was also dealt with in [8] for the case that $L(X) = L(D) = L$ is a constant matrix, so that a, r, and q are constant matrices, too.

We obtain an improved interval Newton operator by the definition

$$NV(X) := \check{x} - S(X)(a(X)\, G(\check{x})), \tag{4.4}$$

where

$$S(X) := [e - q(X), e + q(X)]. \tag{4.5}$$

Because of the subdistributivity the inclusion

$$NV(X) \subseteq N(X) \tag{4.6}$$

is true.

For the midpoint and radius of $NV(X)$ we have the following

LEMMA 4.1. *With the abbreviations* $H := a(X)\, G(\check{x})$ *and* $S_D := (S(X)_{ik}\delta_{ik})$, *the following relations hold:*

$$\operatorname{mid} NV(X) = x - \operatorname{mid}(S_D H), \tag{4.7}$$

$$\operatorname{rad} NV(X) = \operatorname{rad} N(X) - \inf\{\operatorname{rad} S_D \cdot \operatorname{rad} H,$$
$$\operatorname{rad} S_D \cdot |\operatorname{mid} H|, \operatorname{rad} H\}. \tag{4.8}$$

Proof. With the diagonal part S_D of $S(X)$ and the off-diagonal part $S' := (S(X)_{ik}(1 - \delta_{ik}))$ we have the direct splitting $S(X) = S_D \oplus S'$ and

$$\operatorname{mid} S_D = e, \qquad \operatorname{mid} S' = 0. \tag{4.9}$$

By (4.4) and formula (B27) of [10] we get

$$NV(X) = \check{x} - (S_D \oplus S')\, H = \check{x} - S_D H - S'H.$$

By (4.9) and (2.6), mid $S'H = 0$ so that (4.7) follows from (2.4).
By (2.7)–(2.10),

$$\operatorname{rad} NV(X) = \operatorname{rad}(S_D H) + \operatorname{rad} S'\, |H|$$
$$= \operatorname{rad}(S_D H) + (\operatorname{rad} S - \operatorname{rad} S_D)\,|H|.$$

On the other hand, (4.3) implies

$$\text{rad}\, N(X) = \text{rad}\, H + (\text{rad}\, S_D)|H|$$

so that

$$\text{rad}\, N(X) - \text{rad}\, NV(X) = \text{rad}\, H + (\text{rad}\, S_D)|H| - \text{rad}(S_D H).$$

If we apply Proposition 2.1 with $A = S_D$, $X = H$ and observe (4.9) we obtain

$$\begin{aligned}
&\text{rad}\, N(X) - \text{rad}\, NV(X)\\
&\quad = \text{rad}\, H + (\text{rad}\, S_D)|H|\\
&\qquad - \sup\{\text{rad}\, H + \text{rad}\, S_D \cdot |\text{mid}\, H|, \text{rad}\, H + \text{rad}\, S_D \cdot \text{rad}\, H,\\
&\qquad\quad \text{rad}\, S_D \cdot |\text{mid}\, H| + \text{rad}\, S_D \cdot \text{rad}\, H\}\\
&\quad = \inf\{\text{rad}\, S_D(|H| - \text{mid}\, H), \text{rad}\, S_D(|H| - \text{rad}\, H),\\
&\qquad \text{rad}\, H + \text{rad}\, S_D(|H| - |\text{mid}\, H| - \text{rad}\, H)\}.
\end{aligned}$$

By (2.11), this simplifies to (4.8). ∎

Remarks. 1. The infimum in (4.8) is a measure for the improvement of the operator NV over N.

2. If either mid $H = 0$ or rad $H = 0$ then (4.8) implies rad $NV(X) =$ rad $N(X)$ whence $NV(X) = N(X)$ by (4.6).

If G degenerates to a function g and if x^* is a zero of g then $x^* \in X$ implies $x^* \in N(X)$. An analogous property is also valid for the improved operator (4.4).

THEOREM 4.1. *If* $0 \in G(x^*)$ *and* $x^* \in X$ *then*

$$x^* \in NV(X). \tag{4.10}$$

Proof. By assumption (3.2) there exist two matrices $l_1 \in L(X)$ and $l_2 \in L(X)$, so that

$$\begin{aligned}
\underline{g}(\check{x}) - \underline{g}(x^*) &= l_1(\check{x} - x^*),\\
\bar{g}(\check{x}) - \bar{g}(x^*) &= l_2(\check{x} - x^*).
\end{aligned} \tag{4.11}$$

Because of $0 \in G(x^*)$ there exists a nonnegative diagonal matrix d with $\sigma(d) \leqslant 1$, so that the equation

$$d\underline{g}(x^*) + (e - d)\,\bar{g}(x^*) = 0 \tag{4.12}$$

holds.

From (4.11) and (4.12) we obtain

$$g := dg(\check{x}) + (e-d)\,\tilde{g}(\check{x}) = (dl_1 + (e-d)\,l_2)(\check{x} - x^*) \in G(\check{x}),$$

where $l := dl_1 + (e-d)\,l_2 \in dL(X) + (e-d)\,L(X) = L(X)$ (since $d \geqslant 0$ and $e-d \geqslant 0$, the distributive law is valid), and we have

$$x^* = \check{x} - l^{-1}g. \tag{4.13}$$

To include $l^{-1}g$ we define

$$b := a(\check{l} - l) = e - al \qquad \text{(because } a\check{l} = e\text{).} \tag{4.14}$$

Then

$$|b| \leqslant |a|\,|\check{l} - l| \leqslant |a|\,\tfrac{1}{2}(\bar{l} - \underline{l}) = r. \tag{4.15}$$

From (4.14) follows $l^{-1} = (e-b)^{-1}\,a$, so

$$l^{-1}g = ((e-b)^{-1}\,a)\,g = (e-b)^{-1}\,(ag). \tag{4.16}$$

Now we can include $(e-b)^{-1}$. By (4.15) we have $\sigma(b) \leqslant \sigma(r) < 1$ and therefore

$$|(e-b)^{-1} - e| = \left|\sum_{i=1}^{\infty} b^i\right| \leqslant \sum_{i=1}^{\infty} r^i = r(e-r)^{-1} = q \qquad \text{by (3.6).}$$

This implies

$$(e-b)^{-1} \in [e-q, e+q] = S. \tag{4.17}$$

Inserting the relation (4.17) in (4.16) and using $ag \in aG$ we obtain

$$l^{-1}g \in S(aG). \tag{4.18}$$

By (4.4) and (4.13) the incluson (4.10) follows. ▮

Remarks. 1. The application of the associative law in (4.16) is important, because otherwise we must use the inclusion

$$(e-b)^{-1}\,a \in [a - q\,|a|, a + q\,|a|]$$

which yields

$$l^{-1}g \in (Sa)\,G. \tag{4.19}$$

For $n > 1$ this is worse than (4.8) since then $S(aG) \subseteq (Sa)\,G$ and the inclusion is generally proper, even if G degenerates to a function (cf. p. 124

of Alefeld and Herzberger [2]). In particular, the interval Newton operator for function described in Alefeld and Herzberger [3] which is based on (4.19) is inferior to the present one.

2. An operator which possesses the property (4.10) is called *inclusion operator* in [8].

5. Comparison with an Optimal Inclusion

From the proof of Theorem 4.1 we learn that

$$\{l^{-1}g \mid l \in L(X),\ g \in G(\check{x})\} \subseteq S(aG(\check{x})). \tag{5.1}$$

In order to assess the quality of the inclusion (5.1) we need to compare (5.1) with the optimal inclusion interval. This is possible if $G(\check{x}) \geqslant 0$ or $(G(\check{x}) \leqslant 0)$ and L is inverse isotone (i.e., if all $l \in L$ are inverse isotone).

Let $0 \leqslant G(\check{x})$ and let L be inverse isotone. Then

$$\begin{aligned}\{l^{-1}g \mid l \in L(X), g \in G(\check{x})\} &\subseteq [\bar{l}^{-1}, \underline{l}^{-1}]\, G(\check{x}) \\ &= [\bar{l}^{-1}\underline{g}(\check{x}), \underline{l}^{-1}\bar{g}(\check{x})]\end{aligned} \tag{5.2}$$

is an optimal inclusion interval (Beeck [4]). On the other hand it follows from $a = \bar{l}^{-1} \geqslant 0$ and $G(\check{x}) \geqslant 0$ that

$$\sup(S(aG(\check{x})) = (e+q)\, a\bar{g}(\check{x}). \tag{5.3}$$

Inserting $\underline{l}$ in (4.14), b becomes identical to r, so $\underline{l}^{-1} = (e-r)^{-1}\, a$.

Since $(e-r)^{-1} = e + q$ we obtain

$$(e+q)\, a\bar{g}(\check{x}) = \underline{l}^{-1}\bar{g}(\check{x}),$$

i.e., the upper bound of the optimal inclusion (5.2) coincides with the upper bound of the inclusion (5.1). Similarly if $G(\check{x}) \leqslant 0$ and L is inverse isotone then

$$\inf S(aG(x)) = (e+q)\, a\underline{g}(x) = \underline{l}^{-1}\underline{g} = \inf([\bar{l}^{-1}, \underline{l}^{-1}]\, G(\check{x})),$$

i.e., in this special case the lower bounds are identical.

6. Fixinterval Theorems

Let G be a function strip defined in D (see (3.1)) and let N resp. NV denote the interval Newton operator of G with the domain D defined by (4.1) resp. (4.4).

We say that $X \in \mathbb{I}D$ is a *fixinterval* of N resp. of NV, if $X = N(X)$, resp. $X = NV(X)$.

In this section we make some assertions about the uniqueness and existence of fixintervals of N resp. NV.

PROPOSITION 6.1. *X is a fixinterval of N iff X is a fixinterval of NV.*

Proof. 1. From $X = N(X)$ and (4.2) follows $\operatorname{mid} X = \operatorname{mid} N(X) = \check{x} - \operatorname{mid}(a(X)\, G(\check{x}))$, so $\operatorname{mid}(a(X)\, G(\check{x})) = 0$. By Remark 2 after Lemma 4.1 this implies $NV(X) = N(X)$.

2. $X = NV(X)$ implies $\operatorname{mid} X = \operatorname{mid} NV(X)$ and by (4.7) $\operatorname{mid}(S_D \cdot (aG(\check{x}))) = 0$. The application of Proposition 2.2 yields $\operatorname{mid}(aG(\check{x})) = 0$ (because by (4.9) $\operatorname{mid} S_D = e$), so that $NV(X) = N(X)$. ∎

For the following we confine ourselves to the operator N, since by Proposition 6.1 the assertions about fixintervals of the operator N are also valid for those of the operator NV.

PROPOSITION 6.2. *All fixintervals of N have the same midpoint $\check{x}$.*

Proof. Let $X_1^* \subseteq D$ and $X_2^* \subseteq D$ be two different fixintervals of N. Then $\operatorname{mid} G(\check{x}_1^*) = \operatorname{mid} G(\check{x}_2^*) = 0$ follows as in Proposition 6.1 because $a(X)$ is regular.

Applying the Lipschitz condition (3.7) for $X = D$ we obtain

$$|\check{x}_1^* - \check{x}_2^*| \leqslant r(D)\, |\check{x}_1^* - \check{x}_2^*|.$$

Because of $\sigma(r(D)) < 1$ this inequality can be true only if $\check{x}_1^* = \check{x}_2^*$. ∎

If G degenerates to a function g then this implies that there exists at most one fixpoint of N. But in the general case it is possible that N has more than one fixinterval.

EXAMPLE 6.1. Let $\bar{g}(x) := x^3 + 18x - 106$, $\underline{g}(x) := x^3 + 3x - 106$. Then $\operatorname{mid} G(x) = x^3 + 10.5x - 106$, $\operatorname{rad} G(x) = 7.5x$, $L(X) = 3[\underline{x}^2 + 1, \bar{x}^2 + 6]$,

$$a(X) = \frac{2}{3(\underline{x}^2 + \bar{x}^2 + 7)}, \quad r(X) = \frac{\bar{x}^2 - \underline{x}^2 + 5}{\underline{x}^2 + \bar{x}^2 + 7}, \quad (e - r(x))^{-1} = \frac{\underline{x}^2 + \bar{x}^2 + 7}{2(\underline{x}^2 + 1)}.$$

For each interval $D \geqslant 0$ the condition (3.5) is fulfilled. Now suppose that $\check{x} = 4$ so that $\operatorname{mid} G(\check{x}) = 0$, $\operatorname{rad} G(\check{x}) = 30$. From (4.3) and $\operatorname{mid}(aG(\check{x})) = 0$ follows by (2.11) that $\operatorname{rad} N(X) = (e + q)\, a \operatorname{rad} G(\check{x}) = (e - r(X))^{-1}\, a(X) \cdot \operatorname{rad} G(\check{x}) = 10/(\underline{x}^2 + 1)$.

If $X_1^* = [2, 6]$ then $\operatorname{rad} N(X_1^*) = 2 = \operatorname{rad} X_1^*$ and $\operatorname{mid} N(X_1^*) = \operatorname{mid} X_1^* = 4$, so $N(X_1^*) = X_1^*$. If $X_2^* = [3, 5]$ then $\operatorname{rad} N(X_2^*) = 1 = \operatorname{rad} X_2^*$ and $\operatorname{mid} N(X_2^*) = \operatorname{mid} X_2^* = 4$, so $N(X_2^*) = X_2^*$.

So X_1^* and X_2^* are two different fixintervals of N.

A consequence of Proposition 6.2 is the following

THEOREM 6.3. *Let N be defined by (7.1) with $L(X) = L(D) = \text{constant}$. Then the operator N has at most one fixinterval in D.*

Proof. By Proposition 6.2 all fixintervals have the same midpoint $\check{x}$. If a and q are constant then rad $N(X)$ depends only on $\check{x}$; therefore all fixintervals have the same radius too, which means that they are identical. ∎

For the operator defined by (4.1) we write in the following N_0 if $L(X) = L(D)$ for all $X \in \mathbb{I}D$, and N otherwise (i.e., if $L(X)$ is variable).

Generally fixintervals of N_0 (if they exist) need not coincide with fixintervals of N (for the same $G(x)$).

EXAMPLE 6.2. $n = 1$, $D = [1, 10]$, $\bar{g}(x) := x^3 + 46.5x - 290$, $\underline{g}(x) := x^3 + 19.5x - 290$. $L(X) = [3\underline{x}^2 + 19.5, 3\bar{x}^2 + 46.5]$ is a Lipschitz interval, and we have

$$r(X) = \frac{3(\bar{x}^2 - \underline{x}^2) + 27}{3(\underline{x}^2 + \bar{x}^2) + 66} < 1 \qquad \text{for all } x \subseteq D,$$

$$\text{rad } N(X) = (1 - r(X))^{-1}\, a(X) \text{ rad } G(\check{x}) = \frac{27\check{x}}{6\underline{x}^2 + 39}.$$

Since mid $G(\check{x}) = \check{x}^3 + 33\check{x} - 290$ has in D the unique zero $\check{x} = 5$, this is the midpoint of all fixintervals. With $L(D) = [22.5, 346.5]$ we obtain rad $N_0(X) = (1 - r(D))^{-1}\, a(D)$ rad $G(\check{x}) = 3$. Therefore $X_0^* = [2, 8]$ is a fixinterval of N_0. But $X^* := [4, 6]$ is a fixinterval of N, because rad $X^* =$ rad $N(X^*) = (27 \cdot 5)/(6 \cdot 16 + 39) = 1$. This example shows that $X^* \subset X_0^*$. This relation is generally valid:

THEOREM 6.4. *If $X_0^* \subseteq D$ is a fixinterval of N_0 then the relation*

$$X^* \subseteq X_0^* \tag{6.1}$$

holds for each fixinterval X^* of N.

Proof. From (4.3) follows because of $\text{mid}(aG(\check{x}^*)) = 0$, (2.11), and $e + q = (e - r)^{-1}$ that

$$\begin{aligned} \text{rad } X^* &= \text{rad } N(X^*) = (e - r(X^*))^{-1}\, |a(X^*)|\, \text{rad}(G(\check{x}^*)) \\ &\leqslant (e - r(D))^{-1}\, |a(D)|\, \text{rad}(G(\check{x}_0^*)) \\ &= \text{rad } N_0(X_0^*) = \text{rad } X_0^* \end{aligned}$$

(the last inequality is a consequence of Lemma 3.1 and $\check{x}^* = \check{x}_0^*$). ∎

It is even possible that N possesses one or more fixintervals, but N_0 has no fixinterval in D.

EXAMPLE 6.3. In Example 6.1 we have seen that N has two fixintervals $X_1^* = [2, 6]$ and $X_2^* = [3, 5]$ with the common midpoint $\check{x}^* = 4$. If $D = [1, 10]$ and mid $X = 4$ then rad $N_0(X) = 5$ which is independent of rad X. Supposing that N_0 has a fixinterval $X_0^* \subseteq D$ then mid $X_0^* = 4$ and rad $X_0 = 5$ which is a contradiction to $X_0^* \subseteq D$.

But the converse situation is not possible:

THEOREM 6.5. *If $X_0^* \subseteq D$ is a fixinterval of N_0 then N has at least one fixinterval.*

Proof. Let mid $X_0^* = \check{x}_0^*$ and rad $X_0^* = \text{rad } N(X_0^*) = (e - r(D))^{-1} |a(D)| \text{ rad } G(\check{x}_0^*)$. Consider the continuous function $f(\text{rad } X) = (e - r(X))^{-1} |a(X)| \text{ rad } G(\check{x}_0^*)$ for all X with mid $X = \check{x}_0^* =$ constant and $0 \leqslant \text{rad} X \leqslant \text{rad } X_0^*$. By Lemma 3.1, f is monotone in the interval $[0, \text{rad } X_0^*]$, Moreover $f(0) = (e - r(\check{x}_0^*))^{-1} |a(\check{x}_0^*)| \text{ rad } G(\check{x}_0^*) \geqslant 0$ and $f(\text{rad } X_0^*) = (e - r(X_0^*))^{-1} |a(X_0^*)| \text{rad } G(\check{x}_0^*) \leqslant \text{rad } N(X_0^*) = \text{rad } X_0^*$ by Lemma 3.1. Therefore $\{f(\text{rad } X) | \text{rad } X \in [0, \text{rad } X_0^*]\} \subseteq [0, \text{rad } X_0^*]$, and by Brouwer's fixpoint theorem there exists a rad X^* so that rad $X^* = f(\text{rad } X^*) = \text{rad } N(X^*)$. Since the midpoint $\check{x}_0^*$ is also the midpoint of X^* it follows that $N(X^*) = X^*$. ∎

The following existence theorem for a fixinterval of the operator N_0 is at the same time (by Theorem 6.5) an existence theorem for a fixinterval of N (which must not be unique).

THEOREM 6.6. *Let N_0 be defined by* (4.1) *with $L(X) = L(D)$, $a := a(D)$, and $r := r(D)$, and let the condition*

$$\text{rad } D \geqslant (e - r)^{-1} \left(\text{rad}(aG(\check{d})) + (e - r)^{-1} (e + r) \cdot |\text{mid}(aG(\check{d}))|\right) \quad (6.2)$$

be fulfilled. Then N_0 has exactly one fixinterval in D.

Proof. Let $X_0 \subseteq D$ with mid $X_0 = \check{d}$ and

$$X_{k+1} := N_0(X_k), \qquad k := 0, 1, 2, \ldots \quad (6.3)$$

Then from (4.1) follows $\check{x}_{k+1} - \check{x}_k = -\text{mid}(aG(\check{x}_k))$. Applying (3.7) we obtain

$$\begin{aligned} |\text{mid}(aG(\check{x}_k))| &\leqslant r\, |\text{mid}(aG(\check{x}_{k-1}))| \leqslant \cdots \\ &\leqslant r^k\, |\text{mid}(aG(\check{d}))| \end{aligned} \quad (6.4)$$

and

$$|\check{x}_k-\check{d}| \leqslant |\check{x}_k-\check{x}_{k-1}|+|\check{x}_{k-1}-\check{x}_{k-2}|+\cdots+|\check{x}_1-\check{x}_0|$$
$$=|\mathrm{mid}(aG(\check{x}_{k-1}))|+|\mathrm{mid}(aG(\check{x}_{k-2}))|+\cdots+|\mathrm{mid}(aG(\check{d}))|$$
$$\leqslant (r^{k-1}+r^{k-2}+\cdots+e)|\mathrm{mid}(aG(\check{d}))|,$$

so

$$|\check{x}_k-\check{d}| \leqslant (e-r)^{-1}|\mathrm{mid}(aG(\check{d}))| \text{ for all } k\in\mathbb{N}. \tag{6.5}$$

If $X_k \subseteq D$ then from (4.3) and (6.3) follows in view of (2.11) and (3.6)

$$\mathrm{rad}\, X_{k+1}=\mathrm{rad}\, N_0(X_k)=(e-r)^{-1}\left(\mathrm{rad}(aG(\check{x}_k))+r\,|\mathrm{mid}(aG(\check{x}_k))|\right).$$

Inserting (3.8), (6.4), and (6.5) we get

$$\mathrm{rad}\, X_{k+1} \leqslant (e-r)^{-1}\left(\mathrm{rad}(aG(\check{d}))+r(e-r)^{-1}|\mathrm{mid}(aG(\check{d}))| + r^{k+1}|\mathrm{mid}(aG(\check{d}))|\right)$$
$$\leqslant (e-r)^{-1}\left(\mathrm{rad}(aG(\check{d}))+(e-r)^{-1}(e+r)|\mathrm{mid}(aG(\check{d}))|\right) - (e-r)^{-1}|\mathrm{mid}(aG(\check{d}))|$$
$$\leqslant \mathrm{rad}\, D-(e-r)^{-1}|\mathrm{mid}(aG(\check{d}))|$$

(by (6.2) and because $r(e-r)^{-1}+e+r^{k+1} \leqslant (e-r)^{-1}(e+r)$). Considering (6.5) we obtain

$$|\check{x}_{k+1}-\check{d}| \leqslant (e-r)^{-1}|\mathrm{mid}(aG(\check{d}))| \leqslant \mathrm{rad}\, D-\mathrm{rad}\, X_{k+1}.$$

Because of (2.12) the last inequality is equivalent to $X_{k+1} \subseteq D$. Therefore the iteration (6.3) can be carried out for all $k \in \mathbb{N}_0$. By (6.4) we have $|\check{x}_{k+1}-\check{x}_k|=|\mathrm{mid}(aG(\check{x}_k))| \leqslant r^k\, \mathrm{mid}(aG(\check{d}))$. So, since $\sigma(r)<1$,

$$\lim_{k\to\infty} \check{x}_k=\check{x}^*, \qquad \lim_{k\to\infty} (\mathrm{mid}(aG(\check{x}_k)))=\mathrm{mid}(aG(\check{x}^*)),$$

$$\lim_{k\to\infty} (\mathrm{rad}\, X_{k+1})=\lim_{k\to\infty} (\mathrm{rad}\, N_0(X_k))=(e-r)^{-1}\,\mathrm{rad}(aG(\check{d})).$$

This implies $\lim_{k\to\infty} X_k=X^*$, and since N is a continuous function, $X^*=N(X^*)$ holds. By Theorem 6.3, X^* is the unique fixinterval in D. ∎

COROLLARY. *If the condition* (6.2) *is fulfilled then N has at least one fixinterval in D.*

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium