Characterization of a class of distance regular graphs

By *A. Neumaier* at Freiburg

1. Introduction

A *distance regular graph* is a connected graph Γ (finite, undirected, without loops or multiple edges) such that the sets $\Gamma_i(x)$ of vertices (or *points*) at distance i from $x \in \Gamma$ have the following regularity property: For $x, y \in \Gamma$ at distance $d(x, y) = i$,

$$(1) \qquad |\Gamma_1(y) \cap \Gamma_j(x)| = \begin{cases} a_i & \text{if } \; j = i, \\ b_i & \text{if } \; j = i+1, \\ c_i & \text{if } \; j = i-1. \end{cases}$$

The list

$$\iota(\Gamma) := \{b_0, b_1, \ldots, b_{d-1};\; c_1, c_2, \ldots, c_d\},$$

where d is the diameter of Γ, is called the *intersection array* of Γ (Biggs [1]). In particular, Γ is regular of valency $k := b_0$, and

$$(2) \qquad a_0 = c_0 = b_d = 0, \quad c_1 = 1, \quad a_i + b_i + c_i = k \quad (i = 1, \ldots, d).$$

We use $\lambda := a_1$ for the number of common neighbours of two adjacent points, and $\mu := c_2$ for the number of common neighbours of two points at distance two. The number $k_i := |\Gamma_i(x)|$ of points at distance i from a given point x can be recursively computed by

$$(3) \qquad k_0 = 1, \quad k_1 = k, \quad k_{i+1} = \frac{k_i b_i}{c_{i+1}} \quad (i = 0, \ldots, d-1),$$

and the total number of points of Γ is

$$(4) \qquad v = 1 + k_1 + \cdots + k_d.$$

From the spectral theory of distance regular graphs in Biggs [1] it follows that the primitive idempotents E of the adjacency algebra of Γ have entries

$$E_{xy}=\frac{f}{v}u_i \quad \text{for} \quad d(x,y)=i, \tag{5}$$

where $(u_0, u_1, \ldots, u_d)^T$ is an eigenvector of the tridiagonal matrix

$$T(\Gamma)=\begin{pmatrix} 0 & k & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & c_2 & & \cdot & & \\ & & \cdot & & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & \cdot & b_{d-1} \\ 0 & & & & c_d & a_d \end{pmatrix}$$

normalized such that $u_0=1$. Then

$$f:=\left(\sum_{i=0}^{d} k_i u_i^2\right)^{-1} v \tag{6}$$

is the multiplicity of the corresponding eigenvalue; it is always a positive integer.

In this paper we are concerned with distance regular graphs of diameter d with intersection array

$$b_i=\frac{1}{2}(d-i)(a-ci), \quad c_i=i+\binom{i}{2}c \quad (i=0,\ldots,d) \tag{7}$$

for suitable numbers a and c. If $d=2$ these graphs are the strongly regular graphs with smallest eigenvalue -2, determined by Seidel [14]; therefore we assume that $d\geqq 3$. Now relations (2) show that

$$k=\frac{1}{2}ad, \quad \lambda=\frac{1}{2}(a-2+c(d-1)), \quad \mu=c+2, \tag{8}$$

$$a_i=\frac{i}{2}(a-2+c(d+1-2i)); \tag{9}$$

in particular, a and c are integers and

$$a \geqq 2+c(d-1), \quad c \geqq 0 \tag{10}$$

since $b_{d-1} \geqq 1$, $c_3 \geqq 1$. We shall prove in Section 3 that every distance regular graph of diameter $d \geqq 3$ with intersection array (7) is isomorphic to one of the following graphs:

(i) *Hamming graphs* $H(d, n)$: Points are the words of length d over an alphabet of n letters, adjacent if they differ in just one letter.

(ii) *Egawa graphs* $E(d, 4)$: A class of graphs described by Egawa [8] with the same parameters as $H(d, 4)$.

(iii) *Johnson graphs* $J(n, d)$: Points are the d-subsets of an n-set ($n \geqq 2d$), adjacent if they differ in just one element.

(iv) *Half cubes* $D_{n,n}$: Points are the words of even weight and length n over a binary alphabet, adjacent if they differ in just two letters.

(v) *The Gosset graph* Q on 56 vertices, related to the 28 bitangents of a quartic surface: Vertices and edges of the polytope 3_{21} (cf. Coxeter [5]).

The intersection arrays of these graphs are given by (7), where

$$\begin{array}{lll}
c=0,\ a=2(n-1) & \text{for} & H(d, n), \\
c=0,\ a=6 & \text{for} & E(d, 4), \\
c=2,\ a=2(n-d) & \text{for} & J(n, d), \\
c=4,\ a=4d-2 & \text{for} & D_{n,n},\ n=2d, \\
c=4,\ a=4d+2 & \text{for} & D_{n,n},\ n=2d+1, \\
c=8,\ a=18 & \text{for} & Q, \quad d=3.
\end{array}$$

The case $c=0$ has been completely determined by Egawa [8]; therefore we shall assume $c \geqq 1$. Previous work on characterization of the Johnson graphs by (7) with $c=2$ includes work by Dowling [7], Moon [10], [11]; they prove uniqueness for certain pairs (n, d). Here we determine the graphs with (7) by representing the edges as norm 2 vectors in an integral lattice. We then use the relations between root systems and line graphs discovered by Cameron, et. al [4] and some technical results (provided in Section 2) to show that only the graphs mentioned occur.

2. Line graphs and locally line graphs

For our characterization of distance regular graphs with (7) we need preliminary characterizations of certain locally line graphs and certain line graphs. These are provided in this section where, however, a slightly more general situation is treated. First we define the following graphs:

complete graphs K_m with m mutually adjacent vertices;

complete multipartite graphs $K_{m_1,\dots,m_s}$ with $s \geqq 2$ classes of points with sizes $m_1, \dots, m_s$; points are adjacent iff they are in different classes (note $K_{1,\dots,1} = K_s$);

coclique extensions of a graph Δ, i.e. graphs obtained from Δ by replacing the vertices x of Δ by disjoint sets C_x (which may be empty) and joining two points $\bar{x} \in C_x$ and $\bar{y} \in C_y$ iff x and y are joined in Δ;

line graphs $L(\Delta)$ of a graph Δ, its vertices are the edges of Δ, adjacent if they have a common point;

triangular graphs $T(m) = L(K_m)$;

grids $m \times n = L(K_{m,n})$.

We use the notation $x \sim y$ for adjacent vertices x, y and $x \not\sim y$ for nonadjacent distinct vertices x, y. A *clique* is a complete subgraph of Γ. The *neighbourhood* of $x \in \Gamma$ is the set $\Gamma(x) = \Gamma_1(x)$ of all points adjacent with x. Subgraphs induced on $\Gamma(x) \cap \Gamma(y)$ for x, y at distance 2 are called *μ-graphs* of Γ. If $\mathscr{P}$ is a property of graphs, Γ is called *locally $\mathscr{P}$* if for each $x \in \Gamma$, the neighbourhood $\Gamma(x)$ is isomorphic to a graph with property $\mathscr{P}$. We write $\cong$ for graph isomorphism.

Proposition 1. *Let Γ be a connected graph which is locally complete multipartite. Then Γ is either triangle-free or complete multipartite.*

Proof. If Γ is not triangle-free then Γ contains a maximal induced complete multipartite subgraph K with $s \geqq 3$ classes. If $x \in \Gamma \setminus K$ has distance one from K then consideration of $\Gamma(y)$ for the neighbours y of x in K shows that K has neighbours of x in $s-1$ or s classes of K, and that if x is adjacent to one point of a class it is adjacent to all points of the class. But this implies that $K \cup \{x\}$ is complete multipartite. Since K is maximal and Γ is connected, $\Gamma = K$. $\square$

Proposition 2. *Let Γ be a connected graph, locally a line graph of a complete multipartite graph, and suppose that all μ-graphs contain the same number $\mu \leqq 6$ of vertices. Then*:

(i) *If $\mu \neq 5$ then each neighbourhood $\Gamma(z)$, $z \in \Gamma$, is a grid or a triangular graph.*

(ii) *If Γ is locally triangular then $\mu = 6$, and there is an integer m such that Γ is locally $T(m)$.*

(iii) *If Γ is locally a grid then $\mu \in \{4, 6\}$, and there are integers m, n such that Γ is locally $m \times n$.*

Proof. Let H be the μ-graph on $\Gamma(x) \cap \Gamma(y)$ for x, y at distance 2, and let $z \in H$. Then $\Gamma(z) \cong L(\Delta)$ for some complete multipartite graph Δ, and x, y are represented by vertex-disjoint edges $\bar{x}, \bar{y}$ of Δ. The graph induced on the neighbours of z in H (represented by the set of edges joining $\bar{x}$ and $\bar{y}$ in Δ) is a 2-coclique, a $K_{2,1}$, or a quadrangle; hence by Proposition 1, each connected component of H is triangle-free or complete multipartite. Since $|H| \leqq 6$ by assumption, and points of H have valency at least 2, each μ-graph is a quadrangle, a pentagon, a hexagon, a $K_{2,2,1}$, or $K_{2,2,2}$.

Now suppose that $\Gamma(z) \cong L(\Delta)$, where Δ is complete multipartite but neither complete nor bipartite. Then Δ contains a $K_{2,1,1}$, and the points represented by vertex-disjoint edges of a $K_{2,1,1}$ must have $K_{2,2,1}$ as a μ-graph. In particular if $\mu \neq 5$ then all neighbourhoods $\Gamma(z)$ are grids (Δ bipartite) or triangular graphs (Δ complete), so that (i) holds. Next suppose that Γ is locally triangular. Then the graph induced on the neighbours of z in a μ-graph H is a quadrangle so that H is a $K_{2,2,2}$ and $\mu=6$. And if $\Gamma(z) \cong T(m)$, $\Gamma(z') \cong T(m')$, $z \sim z'$, then consideration of the common neighbours of z and z' shows that $m=m'$. Since Γ is connected, Γ is locally $T(m)$, and (ii) holds. Finally suppose that Γ is locally a grid. Then the graph induced on the neighbours of z in H is a 2-coclique so that H is a μ-gon, $\mu \in \{4, 5, 6\}$. But H is contained in the grid $\Gamma(x)$ so that $\mu \neq 5$. Now one shows as before that all neighbourhoods $\Gamma(z)$ are isomorphic. □

A *semibiplane* is a set B of subsets (*blocks*) of a set X of points such that distinct points are in 0 or 2 blocks and distinct blocks intersect in 0 or 2 points.

Proposition 3. *Let Γ be connected and locally $T(m)$. Then:*

(i) *Γ is the point graph of a semibiplane with block size m.*

(ii) *If Γ contains at least 2^{m-1} points then Γ is a half-cube.*

Proof. If $m \leqq 4$ then Γ is locally complete multipartite; hence it is K_2, K_4, or $K_{2,2,2,2}$ and the proposition holds. If $m \geqq 5$ then the local structure of Γ implies that the set of m-cliques as blocks is a semibiplane. Since each point is in m blocks, the incidence graph is a rectagraph (see Neumaier [12]) of valency m, hence has at most 2^m points, with equality iff the rectagraph is a cube. This implies (ii). □

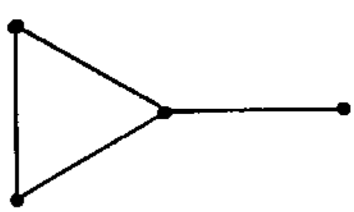
Proposition 4 (Moon [10]). *Let Γ be a connected graph which is locally $m \times n$, $m \leqq n$. If each μ-graph is a quadrangle then either Γ is a Johnson graph $J(m+n, m)$ or $m=n$ and $|\Gamma| < \binom{2m}{m}$. (For details on the exceptions if $m=n$ see* Blokhuis and Brouwer [2].)

Proposition 5. *Let Δ be a connected graph such that each pair of nonadjacent vertices of the line graph $L(\Delta)$ has c or $c+1$ common neighbours. If $L(\Delta)$ is not complete then $c \leqq 4$, and one of the following holds.*

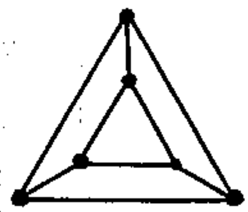
(i) $c=4$ *and* $\Delta = K_m$,

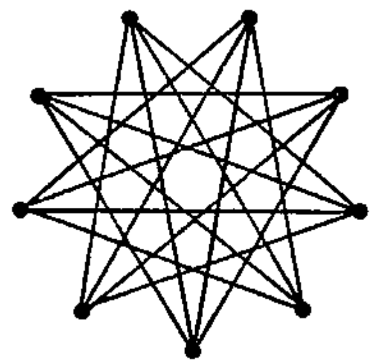
(ii) $c=3$ *and* $\Delta = K_{m,1,\ldots,1}$,

(iii) $c=2$ *and Δ is a coclique extension of the following graph*:



(iv) $c=1$ *and* Δ *is a regular coclique extension of the pentagon, a complete bipartite graph, a triangle, or one of the graphs*



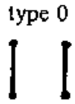


(v) $c=1$ *and* $L(\Delta)$ *is not regular,*

(vi) $c=0$ *and* Δ *has neither triangles nor quadrangles.*

Proof. Two nonadjacent vertices of $L(\Delta)$, i.e. two disjoint edges of Δ, can have at most 4 common neighbours; hence $c \leqq 4$. We consider the possibilities for c when Δ contains prescribed induced subgraphs on 4 vertices containing a pair of disjoint edges:

type 0

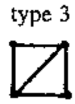


type 1

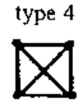
type 2a

type 2b

type 3



type 4



Type i is possible only if $c \in \{i, i-1\}$. Now if $c=0$ then we clearly have (vi). If $c \geqq 2$ then let Δ_0 be a maximal subgraph of Δ of the kind described in the proposition. If $x \in \Delta \setminus \Delta_0$ is a point adjacent to some point of Δ_0 then, in each case, the maximality of Δ_0 implies that Δ contains a forbidden subgraph on four vertices. This contradiction implies $\Delta = \Delta_0$ since Δ is connected. Finally, if $c=1$ and we are not in case (v) then $L(\Delta)$ is regular.

If Δ is bipartite with bipartite classes X_1 and X_2 then points of X_1 must have the same valency s_1+1, and points of X_2 must have the same valency s_2+1. If $s_1=0$ or $s_2=0$ then $\Delta = K_{m,1}$. Otherwise, the set of points at distance 1 from an edge

$$x_1, x_2 \ (x_1 \in X_1, \ x_2 \in X_2)$$

contains $s_{3-i}>0$ points in X_i $(i=1,2)$. A point $\bar{x}_i \in X_i$ at distance 2 from x_1x_2 can have valency at most s_i since disjoint edges are forbidden. Hence this cannot occur, points of X_1 and X_2 are mutually adjacent, and $\Delta = K_{m,n}$ since Δ is connected.

If Δ is not bipartite then, since $L(\Delta)$ is regular, Δ itself is regular. If Δ contains no triangle then one easily sees that Δ is a coclique extension of a pentagon. So let Δ contain some triangle xyz, and let A_u be the set of points adjacent to the point $u \in \{x, y, z\}$ but not to the other points of the triangle. Since types 0, 3, and 4 are forbidden, A_x, A_y, A_z are cocliques of the same size and $\Delta = \{x, y, z\} \cup A_x \cup A_y \cup A_z$. Moreover, each point of A_z is adjacent to at least one point of each edge in $A_x \cup A_y$ (and similarly for any permutation of x, y, z). Now let s be the maximal number such that a point $w \in \Delta \setminus \{x, y, z\}$ has s neighbours in some A_u. If $s \leqq 2$ then Δ must be one of the graphs under (iv); so we assume that $s \geqq 3$. W.l.o.g., let $w \in A_x$ have s neighbours in A_y, and t neighbours in A_z. Then Δ has valency $s+t+1$, and $|A_x| = |A_y| = |A_z| = s+t-1$; in particular $t \geqq 1$. Any point $w' \in A_z$ nonadjacent with w is adjacent with all points of $\Delta(w) \cap A_y$. Now suppose that there is a point $w_0 \in A_y \setminus \Delta(w)$. A neighbour w_1 of w_0 cannot be adjacent to both w and w', hence is adjacent to all s points of $\Delta(w) \cap A_y$. But then w_1 has $\geqq s+1$ neighbours in A_y, contradicting the maximality of s. Hence $A_y \subseteq \Delta(w)$, and $t=1$. Let v be the unique neighbour of w in A_z, and let v' be the unique neighbour of v in A_x. Then each point of A_y is adjacent to y and the $2s-2 \geqq s+1$ points of $A_x \cup A_z \setminus \{v, v'\}$, and since Δ has valency $s+2$, v and v' cannot have neighbours in A_y. But then v has valency $\leqq s+1$, contradiction. □

3. The characterization

In the following, Γ always denotes a distance regular graph with diameter $d \geqq 3$, intersection array (7), and $c \geqq 1$. Let G be the $v \times v$-matrix, indexed by points of Γ, whose (x, y)-entry is

$$G_{xy} = \frac{k}{\lambda+2} - d(x, y). \tag{11}$$

It is easy to check that for the intersection array (7), the vector $(u_0, \ldots, u_d)^T$ with $u_i = 1 - \frac{\lambda+2}{k} i$ $(i = 0, \ldots, d)$ is an eigenvector of $T(\Gamma)$ corresponding to the eigenvalue $k-2-\lambda$. Since $G_{xy} = \frac{k}{\lambda+2} u_i$ if $d(x, y) = i$, G is a positive multiple of the corresponding primitive idempotent; in particular, G is positive semidefinite of rank f given by (6), and G has zero row sums. Therefore, G may considered as the Gram matrix of a set $\bar{\Gamma} = \{\bar{x} | x \in \Gamma\}$ of vectors spanning $\mathbb{R}^f$ (in fact of a spherical 2-design, see [6]) such that

$$(\bar{x}, \bar{y}) = \frac{k}{\lambda+2} - d(x, y) \tag{12}$$

and

$$\sum_{x \in \Gamma} \bar{x} = 0. \tag{13}$$

Now let $x_1, x_2 y_1, y_2 \in \Gamma$. Then by (12),

$$(14) \qquad (\bar{x}_1 - \bar{y}_1, \bar{x}_2 - \bar{y}_2) = d(x_1, y_2) + d(y_1, x_2) - d(x_1, x_2) - d(y_1, y_2)$$

is integral. Therefore the vectors $\bar{x} - \bar{y}$ $(x, y \in \Gamma)$ span a lattice L. Moreover

$$(15) \qquad \bar{x} \in \frac{1}{v} L$$

for all $x \in \Gamma$ since by (13), $v\bar{x} = \sum_{y \in \Gamma} (\bar{x} - \bar{y}) \in L$. In particular, L spans $\mathbb{R}^f$.

Proposition 6. (i) *If $y_1, y_2 \in \Gamma(x)$, $y_1 \neq y_2$ then y_1 and y_2 have c or $c+1$ common neighbours in $\Gamma(x)$. In particular, $\Gamma(x)$ is a regular connected graph of diameter* 2.

(ii) *Each μ-graph of Γ is complete multipartite with classes of size* 1 *or* 2.

Proof. (i) Let z be a common neighbour of y_1 and y_2 which is nonadjacent to x. Then by (14), $a = \bar{y}_1 - \bar{x}$ and $b = \bar{z} - \bar{y}_2$ have $(a, a) = (b, b) = (a, b) = 2$ whence

$$(a-b, a-b) = 0, \quad a = b, \quad \bar{z} = \bar{y}_1 + \bar{y}_2 - \bar{x}.$$

Therefore there is at most one such point z, and the remaining c or $c+1$ points adjacent to y_1 and y_2 are adjacent to x.

(ii) If a μ-graph $\Gamma(y_1) \cap \Gamma(y_2)$ is not as claimed it contains three points x, z, z' such that x is nonadjacent to z and z'; but this configuration contradicts the uniqueness of z in (i). □

Proposition 7. *The lattice L is an irreducible root lattice, i.e. L is isomorphic to one of the lattices*

$$A_n = \{x \in \mathbb{Z}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\},$$
$$D_n = \{x \in \mathbb{Z}^n \mid x_1 + \cdots + x_n \ \textit{even}\},$$
$$E_8 = \left\langle D_8, \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)\right\rangle,$$
$$E_7 = \{x \in E_8 \mid x_1 + \cdots + x_8 = 0\},$$
$$E_6 = \{x \in E_8 \mid x_1 + \cdots + x_6 = x_7 + x_8 = 0\}.$$

Proof. Let $x, y \in \Gamma$. Since Γ is connected there is a path $y = x_0 \sim x_1 \sim \cdots \sim x_n = x$ and $\bar{x} - \bar{y} = \sum_{j=1}^{n} (\bar{x}_i - \bar{x}_{i-1})$. Therefore L is generated by vectors $\bar{x} - \bar{y}$ with $x \sim y$. But by (14) such vectors have norm $(\bar{x} - \bar{y}, \bar{x} - \bar{y}) = 2$. Now suppose that L is the direct sum of two lattices L_1 and L_2. If we call an edge xy of type i if the norm 2 vector $\bar{x} - \bar{y}$ is in L_i, each edge is either of type 1 or type 2. If xyz is a triangle then $(\bar{x} - \bar{y}, \bar{x} - \bar{z}) = 1$ by (14) so that edges in the same triangle have the same type. Since Γ is connected and locally connected (Proposition 6) this implies that all edges have the same type, whence $L = L_1$ or $L = L_2$. Therefore L is irreducible. But by Neumaier [13] (or indirectly from Witt [18]), an irreducible lattice generated by norm two vectors is one of the above lattices. □

This proposition allows us now to make use of the relations between line graphs and root systems discovered by Cameron, et al. [4].

Proposition 8. *Either Γ is locally a line graph, or $L=E_f$, $f\leqq 8$, $d\leqq 4$, and one of the following holds*:

(A) $d=3$, $a=2c+2$, $c\leqq 8$,

$$\iota(\Gamma)=\{3c+3, c+2, 1; 1, c+2, 3c+3\}, \qquad (\bar{x}, \bar{y})=\frac{3}{2}-d(x, y),$$

(B) $d=3$, $a=4c+4$, $c\leqq 3$,

$$\iota(\Gamma)=\{6c+6, 3c+4, c+2; 1, c+2, 3c+3\}, \qquad (\bar{x}, \bar{y})=2-d(x, y),$$

(C) $d=4$, $a=3c+2$, $c\leqq 4$,

$$\iota(\Gamma)=\{6c+4, 3c+3, c+2, 1; 1, c+2, 3c+3, 6c+4\}, \qquad (\bar{x}, \bar{y})=2-d(x, y).$$

Proof. The set of vectors $\{\bar{y}-\bar{x} \mid y\in\Gamma(x)\}$ has Gram matrix $2I+A$, where A is the adjacency matrix of $\Gamma(x)$. Hence $\Gamma(x)$ has smallest eigenvalue $\geqq -2$. But Cameron et al. [4] and Bussemaker et al. [3] show that a regular, connected graph with smallest eigenvalue $\geqq -2$ is either a $K_{2,2,\ldots,2}$ (but in this case $b_1=1$ and hence $d=2$), or a line graph, or a graph with k points and valency λ related by

$$\frac{k}{\lambda+2}=r\in\left\{2, \frac{3}{2}, \frac{4}{3}\right\}, \quad k\leqq 28, \tag{16}$$

cf. [3], Proposition 5.10 and [4], Theorem 4.4. Moreover, in the latter case the corresponding norm 2 vectors span E_8, E_7, or E_6. Hence, if Γ is not locally a line graph, then $L=E_f$, $f\leqq 8$, and by (12) and (16),

$$(\bar{x}, \bar{y})=r-d(x, y) \quad \text{for} \quad x, y\in\Gamma. \tag{17}$$

Since $(\bar{x}, \bar{x})=r$, this implies that $r-d\geqq -r$ and therefore $d\leqq 2r\leqq 4$. If $d=3$ then we must have $r\in\left\{\frac{3}{2}, 2\right\}$, and $k=r(\lambda+2)$ yields the first two cases; if $d=4$ then we have $r=2$ and get the third case (in each case, $k\leqq 28$ yields the restriction on c). □

Proposition 9. *If $L=A_n$ or D_n then Γ is a Johnson graph or a half cube.*

Proof. By Propositions 6 (i) and 8, each $\Gamma(x)$ is a connected regular line graph satisfying the hypothesis of Proposition 5 with $c>0$. Therefore $c\in\{1, 2, 3, 4\}$.

If $c=4$ then $\Gamma(x)\cong L(K_m)$ for some m and Proposition 2 (ii) implies that Γ is locally $T(m)$. Hence $k=\binom{m}{2}$, $\lambda=2(m-2)$ and (8) implies $ad=m(m-1)$, $a+4d=4m-2$. This has the solutions $(a, a)=\left(\frac{n}{2}, 2m-2\right)$ and $(a, d)=\left(\frac{m-1}{2}, 2m\right)$. Hence Γ has the same parameters as the half cube $D_{m,m}$; in particular $v=2^{m-1}$ so that Γ is a half cube by Proposition 3.

If $c=3$ then $\Gamma(x)\cong L(K_m)$ for some m, but $\mu=5$ contradicts Proposition 2 (ii). Hence this case is impossible.

If $c=2$ then $\Gamma(x) \cong L(K_{m,n})$ or $L(K_{n,n,n})$; hence Proposition 2 implies that Γ is locally $m \times n$ $(m \leqq n)$. As before, Γ has the same parameters as the Johnson graph $J(m+n, m)$, and since $\mu=4$, Proposition 4 shows that Γ is a Johnson graph.

If $c=1$ then $\mu=3$. For $x \in \Gamma$, let $\Gamma(x)$ be isomorphic to $L(\Delta_x)$. If some Δ_x contains a triangle then by Proposition 5 (iv), $L(\Delta_x)$ has k points and valency λ, where $(k, \lambda) \in \{(3,2), (9,4), (36,6)\}$; but this conflicts with (8) and (10) since $d \geqq 3$. Hence no Δ_x contains a triangle and no $\Delta(x)$ contains a $K_{2,1,1}$. Now choose $\infty \in \Gamma$. Since $L \subseteq \mathbb{Z}^{n+1}$, an easy induction argument shows that we can choose the basis of $\mathbb{Z}^{n+1}$ in such a way that for all $x \in \Gamma$, $\bar{x} - \bar{\infty}$ is a (0, 1)-vector with precisely $2d(x, \infty)$ nonzero entries. Let $x \in \Gamma_3(\infty)$, and consider the code on the 6 nonzero entries of x. We see that two nonadjacent points in $A = \Gamma(\infty) \cap \Gamma_2(x)$ must have 1 or 2 common neighbours in A, and two nonadjacent points in $B = \Gamma_2(\infty) \cap \Gamma(x)$ must have 1 or 2 common neighbours in B; moreover $|A| = c_3 = |B|$. An attempt to construct A and B now leads to a contradiction. □

By considering the cases left open in Proposition 8 we now obtain our main result.

Theorem. *Let Γ be a distance regular graph with diameter $d \geqq 3$ and intersection array*

$$b_i = \frac{1}{2}(d-i)(a-ci), \quad c_i = i + \binom{i}{2} c \quad (i=0, \ldots, d).$$

Then Γ is isomorphic to one of the graphs $H(d, n)$, $E(d, 4)$, $J(n, d)$, $D_{n,n}$, or Q, described in Section 1.

Proof. By Proposition 9, the only possible exceptions arise with (A), (B), (C) of Proposition 8. In case (A), a result of Taylor and Levingston [16] shows that Γ is a doubled twograph, and then Taylor [15] shows that there are unique solutions for $c=2, 4, 8$ (and no others). The resulting graphs are therefore isomorphic to $J(6, 3)$, $D_{6,6}$, and Q which realize these parameters.

In case (B) we get from (3) and (6) a nonintegral value $f = \frac{22}{3}$ for $c=1$, and a nonintegral value k_2 for $c=3$. For $c=2$, the parameters are those of $J(9, 3)$ and by Moon [10], the only possibility is $J(9, 3)$ itself.

In case (C) we have $r=2$, and by (17), $\bar{\Gamma}$ itself spans a lattice containing E_f and hence coinciding with E_f; therefore Γ is embedded into E_f as a set of norm 2 vectors with $d(\bar{x}, \bar{y}) = 2 - (x, y)$. Now if $c=1$ then the graph obtained from Γ by identifying antipodal pairs is strongly regular and has the parameters of $T(7)$; hence it is $T(7)$ (see e.g. [14]). But $T(7)$ is locally a Petersen graph; hence the same holds for Γ. But there is no locally Petersen graph with the required 42 vertices (Hall [9]). If $c=3$ then k_2 is nonintegral and there is no graph. The remaining possibilities $c=2$ and $c=4$ are more difficult.

Take now $c=2$, then (3) and (6) give $f=7$ so that Γ is embedded into E_7. Each $\Gamma(x)$ is a graph with 16 vertices and valency 6 in E_7, and by the tables of Bussemaker et al. [3], $\Gamma(x)$ is either the Shrikhande graph (No. 69) or a line graph $L(K)$ of a graph K with $\leqq 8$ vertices. In the latter case, K must have 8 points and valency 4; hence $K = K_{4,4}$ since otherwise the vectors corresponding to $\Gamma(x)$ span $D_8 \nsubseteq E_7$. For both admissible local graphs two nonadjacent points in $\Gamma(x)$ have just two common neighbours in $\Gamma(x)$, and hence $(\mu=4)$ a unique neighbour in $\Gamma_2(x)$. Now $|\Gamma_2(x)| = k_2 = 36$

and $\Gamma(x)$ has 72 unordered pairs of nonadjacent points y_1, y_2. Hence by Proposition 6 (ii), a point of $\Gamma_2(x)$ is determined by precisely two pairs of nonadjacent points, the μ-graphs of Γ are quadrangles, and the two common neighbours in $\Gamma(x)$ of y_1, y_2 are always nonadjacent. But this is the case only if $\Gamma(x) \cong L(K_{4,4}) = 4 \times 4$; therefore Γ is locally 4×4 and hence $J(8, 4)$.

Finally, if $c=4$ then each $\Gamma(x)$ is a graph with 28 vertices and valency 12 in E_8, and by [3] again, $\Gamma(x)$ is either a Chang graph (No. 161, 162, 163), or a line graph of a graph K with at most 9 points; clearly $K = K_8$. In both cases two nonadjacent points in $\Gamma(x)$ have just four common neighbours in $\Gamma(x)$, and hence ($\mu=6$) a unique neighbour in $\Gamma_2(x)$. Now $|\Gamma_2(x)| = k_2 = 70$ and $\Gamma(x)$ has 210 unordered pairs of nonadjacent points y_1, y_2. Hence by Proposition 6 (ii), each μ-graph of Γ is a $K_{2,2,2}$, and the four common neighbours in $\Gamma(x)$ of y_1, y_2 must form a quadrangle. But this is the case only if $\Gamma(x) \cong L(K_8) = T(8)$; therefore Γ is locally $T(8)$ and hence $D_{8,8}$. □

Remark. After preparation of the manuscript I was informed by R. A. Liebler that also Paul Terwilliger [17] recently proved the uniqueness of all Johnson graphs $J(n, d)$, $d \geqq 3$.

References

[1] *N. Biggs*, Algebraic graph theory, Cambridge 1974.

[2] *A. Blokhuis* and *A. E. Brouwer*, Locally 4-by-4 grid graphs, Report PM-R 8401, Centrum Wisk. Inform., Amsterdam 1984.

[3] *F. C. Bussemaker, D. M. Cvetković* and *J. J. Seidel*, Graphs related to exceptional root systems, T. H.-Report 76-WSK-05, Eindhoven 1976.

[4] *P. J. Cameron, J. M. Goethals, J. J. Seidel* and *E. E. Shult*, Line graphs, root systems, and elliptic geometry, J. Algebra **43** (1976), 305—327.

[5] *H. S. M. Coxeter*, Extreme forms, Canad. J. Math. **3** (1951), 391—441.

[6] *P. Delsarte, J. M. Goethals* and *J. J. Seidel*, Spherical codes and designs, Geometriae Dedicata **6** (1977), 363—388.

[7] *T. A. Dowling*, A characterization of the T_m graph, J. Combin. Theory **6** (1969), 251—263.

[8] *Y. Egawa*, Characterization of $H(n, q)$ by the parameters, J. Combin. Theory (A) **31** (1981), 108—125.

[9] *J. I. Hall*, Locally Petersen graphs, J. Graph Theory **4** (1980), 173—187.

[10] *A. Moon*, Characterization of the graphs of the Johnson schemes $G(3k, k)$ and $G(3k+1, k)$, J. Combin. Theory (B) **33** (1982), 213—221.

[11] *A. Moon*, On the uniqueness of the graphs $G(n, k)$ of the Johnson schemes, J. Combin. Theory (B) **33** (1982), 256—264.

[12] *A. Neumaier*, Rectagraphs, diagrams, and Suzuki's sporadic simple group, Ann. Discr. Math. **15** (1982), 305—318.

[13] *A. Neumaier*, Lattices of simplex type, SIAM J. Alg. Disc. Meth. **4** (1983), 145—160.

[14] *J. J. Seidel*, Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3, Lin. Algebra Appl. **1** (1968), 281—298.

[15] *D. E. Taylor*, Regular 2-graphs, Proc. London Math. Soc. (3) **34** (1977), 257—274.

[16] *D. E. Taylor* and *R. Levingston*, Distance regular graphs, in: Combinatorial Mathematics (D. A. Holton and J. Seberry, ed.) Lecture Notes in Math. **686**, Berlin-Heidelberg-New York 1978, 313—323.

[17] *P. Terwilliger*, The Johnson graph $J(d, r)$ is unique if $(d, r) \neq (2, 8)$, submitted to Discrete Math.

[18] *E. Witt*, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Abh. Math. Sem. Hamburg **14** (1941), 289—322.

Institut für Angewandte Mathematik, Universität Freiburg i. Br., D-7800 Freiburg i. Br.

Eingegangen 25. Juni 1984