

## The extremal case of some matrix inequalities

By

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**Introduction.** The paper discusses the problem of characterizing the matrices satisfying one of the equalities  $|AB| = |A||B|$ ,  $|(I - A)^{-1}| = (I - |A|)^{-1}$ , or  $|A^{-1}| = \langle A \rangle^{-1}$ , where  $|\cdot|$  denotes componentwise absolute value, and  $\langle A \rangle$  is Ostrowski's comparison matrix for  $A$ . Some open problems remain.

**1. The equation  $|AB| = |A||B|$ .** Denote by  $\mathbb{R}^{n \times m}$  and  $\mathbb{C}^{n \times m}$  the set of real and complex  $n \times m$ -matrices respectively, and denote the  $(i, k)$ -entry of  $A \in \mathbb{C}^{n \times m}$  by  $A_{ik}$ . The identity matrix (of any size) is written as  $I$ . We define the relations  $\varrho \in \{<, \leq, >, \geq\}$  on  $\mathbb{R}^{n \times m}$  by the convention

$$A \varrho B : \Leftrightarrow A_{ik} \varrho B_{ik} \text{ for all } i, k.$$

We consider the *absolute value*  $|\cdot|: \mathbb{C}^{n \times m} \rightarrow \mathbb{R}^{n \times m}$  defined by

$$|A|_{ik} := |A_{ik}| \text{ for all } i, k,$$

and Ostrowski's [3] *comparison operator*  $\langle \cdot \rangle: \mathbb{C}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  defined by

$$\langle A \rangle_{ii} := |A|_{ii}, \quad \langle A \rangle_{ik} := -|A_{ik}| \text{ for } i \neq k.$$

We do not use the more familiar notation  $\mathfrak{M}(A)$  for the comparison matrix of  $A$  in order to emphasise the close relationship to the absolute value apparent from the following easily established relations (cf. [2]).

$$\begin{aligned} |A \pm B| &\leq |A| + |B|, & \langle A \pm B \rangle &\geq \langle A \rangle - |B|, \\ |AB| &\leq |A||B|, & |AB| &\geq \langle A \rangle |B|. \end{aligned}$$

In this section we consider the case of equality in one of these inequalities: the equation  $|AB| = |A||B|$ . As a tool we use the complex sign function  $\text{sgn}: \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$\text{sgn } 0 = 0, \quad \text{sgn } x = x/|x| \text{ if } x \neq 0,$$

and the following obvious observation.

**Lemma.** If  $|\sum x_i| = \sum |x_i|$  then there is a nonzero  $\gamma \in \mathbb{C}$  such that  $\text{sgn } x_i \in \{0, \gamma\}$  for all  $i$ .  $\square$

**Theorem 1.** Let  $A \in \mathbb{C}^{p \times n}$ ,  $B \in \mathbb{C}^{n \times m}$  satisfy the equation

$$(1) \quad |AB| = |A||B|.$$

Then one of the following holds:

- (i) Every row of  $A$  contains some zero entry.
- (ii) Every column of  $B$  contains some zero entry.
- (iii) There are nonsingular diagonal matrices  $D, D', D''$  such that  $D'AD$  and  $D^{-1}BD''$  are real and nonnegative.

*Proof.* The assumption implies that

$$(2) \quad \left| \sum_{j=1}^n A_{ij} B_{jk} \right| = |(AB)_{ik}| = |AB|_{ik} = (|A||B|)_{ik} = \sum_{j=1}^n |A_{ij}| |B_{jk}|,$$

for  $i = 1, \dots, p$  and  $k = 1, \dots, m$ . With

$$(3) \quad \alpha_{ij} := \operatorname{sgn} A_{ij}, \quad \beta_{jk} := \operatorname{sgn} B_{jk},$$

(2) and the lemma imply the existence of nonzero numbers  $\gamma_{ik} \in \mathbb{C}$  with

$$(4) \quad \alpha_{ij} \beta_{jk} \in \{0, \gamma_{ik}\} \quad \text{for all } i, j, k.$$

Now suppose that neither (i) nor (ii) holds. Then there are indices  $r, s$  such that  $|A_{rj}| |B_{js}| \neq 0$  for  $j = 1, \dots, n$ , and by (4),

$$(5) \quad \alpha_{rj} \beta_{js} = \gamma_{rs} \neq 0.$$

Therefore, the diagonal matrices

$$D' := \operatorname{Diag} \left( \frac{1}{\gamma_{is}} \right), \quad D := \operatorname{Diag}(\beta_{js}), \quad D'' := \operatorname{Diag} \left( \frac{\gamma_{rs}}{\gamma_{rk}} \right)$$

are nonsingular, and by (3) and (5) we have

$$(D'AD)_{ij} = \frac{A_{ij} \beta_{js}}{\gamma_{is}} = |A_{ij}| \frac{\alpha_{ij} \beta_{js}}{\gamma_{is}} \in \{0, |A_{ij}|\},$$

$$(D^{-1}BD'')_{jk} = \frac{B_{jk} \gamma_{rs}}{\beta_{js} \gamma_{rk}} = \frac{\alpha_{rj} B_{jk}}{\gamma_{rk}} = \frac{\alpha_{rj} \beta_{jk}}{\gamma_{rk}} |B_{jk}| \in \{0, |B_{jk}|\}.$$

Hence (iii) holds.  $\square$

Conversely, it is obvious that  $|AB| = |A||B|$  for all pairs  $(A, B)$  satisfying (iii). On the other hand, the characterization of those pairs  $(A, B)$  with  $|AB| = |A||B|$  satisfying (i) or (ii) depends on the zero structure of  $A$  and  $B$  and seems to be a nontrivial combinatorial problem. Simple examples which may occur here are the matrix pairs of type

$$A = (C \ 0), \quad B = \begin{pmatrix} 0 \\ D \end{pmatrix}, \quad \text{or} \quad A = (C \ I), \quad B = \begin{pmatrix} I \\ C \end{pmatrix},$$

with arbitrary  $C, D$  of the proper size. Moreover, with  $(A, B)$ , the pairs  $(M'AM, M^{-1}BM'')$  occur, where  $M, M', M''$  are nonsingular monomial matrices, i.e. have precisely one nonzero entry in each row and column. A particular case, used in the next section, is

$$(6) \quad |AB| = |A||B| \quad \text{if } A \text{ or } B \text{ is diagonal.}$$

2. The equation  $|A^{-1}| = \langle A \rangle^{-1}$ . A square matrix  $A \in \mathbb{C}^{n \times n}$  is called an *H-matrix* if there is a positive vector  $u \in \mathbb{R}^n$  such that  $\langle A \rangle u > 0$ , and it is called an *M-matrix* if, in addition,  $A = \langle A \rangle$ . For equivalent definitions, see e.g. Bermann and Plemmons [1]. The spectral radius of a square matrix  $A \in \mathbb{C}^{n \times n}$  is denoted by  $\sigma(A)$ . In this section we consider equality in Ostrowski's [3] relation  $|A^{-1}| \leq \langle A \rangle^{-1}$  for *H-matrices*, and prove as a preparation the following result of independent interest.

**Theorem 2.** *Let  $A \in \mathbb{C}^{n \times n}$  be a square matrix with  $\sigma(|A|) < 1$ . Then*

$$|(I - A)^{-1}| \leq (I - |A|)^{-1}.$$

Moreover, if  $A$  is irreducible then the following assertions are equivalent:

- (i)  $|(I - A)^{-1}| = (I - |A|)^{-1}$ ,
- (ii)  $|(I - A)^{-1}|$  and  $(I - |A|)^{-1}$  have a common entry.
- (iii)  $A$  is a diagonally similar to a real nonnegative matrix, i.e. a nonsingular diagonal matrix  $D$  exists such that  $D^{-1}AD$  is real and nonnegative.

**Proof.** Since  $\sigma(A) \leq \sigma(|A|) < 1$ , the Neumann series

$$\begin{aligned} (I - A)^{-1} &= I + A + A^2 + \dots + A^l + \dots, \\ (I - |A|)^{-1} &= I + |A| + |A|^2 + \dots + |A|^l + \dots, \end{aligned}$$

converge, and by the triangle inequality and repeated application of  $|AB| \leq |A||B|$  we get (7). - To prove the equivalence of (i), (ii), and (iii), we remark first that (i)  $\Rightarrow$  (ii) obviously.

Now suppose that (ii) holds, where the common entry is in place  $(i, k)$ . The  $(i, k)$  entry of  $C^l$  for a matrix  $C \in \mathbb{C}^{n \times n}$  is  $\sum c_{i_1 i_1} c_{i_1 i_2} \dots c_{i_{l-1} i_l} c_{i_l k}$ , summed over all combinations of indices  $i_1, \dots, i_{l-1}$ . Hence the  $(i, k)$ -entry of  $(I - A)^{-1}$  and  $(I - |A|)^{-1}$  is

$$1 + \sum A_{ii_1} A_{i_1 i_2} \dots A_{i_{l-1} i_l} \quad \text{and} \quad 1 + \sum |A_{ii_1} \dots A_{i_{l-1} i_l}|,$$

respectively, so that by the lemma,  $A_{ii_1} A_{i_1 i_2} \dots A_{i_{l-1} i_l} \geq 0$  for all index sequences  $i, \dots, i_l$  ( $l \geq 0$ ). Now let  $i_0, \dots, i_s$  ( $s \geq 0$ ) be an arbitrary index sequence. Since  $A$  is irreducible, there are sequences  $j_1, \dots, j_p$  and  $k_1, \dots, k_q$  such that

$$A_{i_0 j_1} \dots A_{j_p i_0} \neq 0, \quad A_{i_0 k_1} \dots A_{k_q i_0} \neq 0.$$

Therefore

$$\begin{aligned} A_{i_0 j_1} \dots A_{j_p i_0} A_{i_0 k_1} \dots A_{k_q i_0} &> 0, \\ A_{i_0 j_1} \dots A_{j_p i_0} A_{i_0 i_1} \dots A_{i_{s-1} i_s} A_{i_s k_1} \dots A_{k_q i_s} &\geq 0, \end{aligned}$$

and therefore, the quotient of these expressions satisfies

$$(8) \quad A_{i_0 i_1} A_{i_1 i_2} \dots A_{i_s i_0} \geq 0$$

for all index sequences  $i_0, i_1, \dots, i_s$  ( $s \geq 0$ ). Let  $S$  be a maximal set of indices such that there exist nonzero numbers  $d_i$  ( $i \in S$ ) such that

$$d_i^{-1} A_{i_1 i_2} \dots A_{i_s i_1} d_{i_1} \geq 0$$

for all index sequences  $i_1, \dots, i_s$  ( $s \geq 0$ ) and all  $i, k \in S$ . By (8),  $S$  is nonempty. If  $S \neq \{1, \dots, n\}$  then, since  $A$  is irreducible, one of the following two cases occurs.

Case 1. There is  $i \notin S$  such that  $A_{ii} \neq 0$  for some  $i \in S$ . Fix one such  $i \in S$  and put  $d_i := d_i/A_{ii}$ . Then for  $k \in S$ , we have

$$d_i^{-1} A_{ii} \dots A_{i_k} d_k = d_i^{-1} A_{ii} A_{ii} \dots A_{i_k} d_k \geq 0.$$

Moreover, since  $A$  is irreducible, there are indices  $l_1, \dots, l_s$  such that  $A_{il_1}, \dots, A_{i_s i} \neq 0$  so that by (8),

$$d_k^{-1} A_{ki} \dots A_{i_s i} d_i = \frac{d_k^{-1} A_{ki} \dots A_{i_s i} A_{ii} \dots A_{i_s i} d_i}{A_{ii} A_{ii} \dots A_{i_s i}} \geq 0.$$

Together with (8) for  $i_0 = i$ , this contradicts the maximality of  $S$ .

Case 2. There is  $i \notin S$  such that  $A_{ii} \neq 0$  for some  $i \in S$ . This leads to a contradiction by the dual argument.

Therefore,  $S = \{1, \dots, n\}$ , and with  $D := \text{Diag}(d_i)$  we have  $(D^{-1}AD)_{ik} = d_i^{-1} A_{ik} d_k \geq 0$  for all  $i, k$  so that (iii) holds. Finally, if (iii) holds then the nonnegative matrix  $B := D^{-1}AD$  is similar to  $A$ , hence  $\sigma(B) < 1$ ,  $(I - B)^{-1} \geq 0$ , and  $|(I - A)^{-1}| = |D(I - B)^{-1}D^{-1}| = |D|(I - B)^{-1}|D|^{-1} = |D|(I - B)^{-1}|D|^{-1} = (I - |D|B|D|^{-1})^{-1} = (I - |A|)^{-1}$  by (6). Therefore (i) holds.  $\square$

**Corollary.** *If  $A \in \mathbb{C}^{n \times n}$  is an irreducible matrix with  $\sigma(|A|) < 1$  not satisfying (iii) then*

$$(7a) \quad |(I - A)^{-1}| < (I - |A|)^{-1}. \quad \square$$

If  $A$  is reducible then the three conditions are no longer equivalent; we only have the implications (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (ii). Counterexamples to (i)  $\Rightarrow$  (iii) are the matrices of shape  $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  which have

$$A^2 = |A|^2 = 0, \quad |(I - A)^{-1}| = |I + A| = I + |A| = (I - |A|)^{-1},$$

and counterexamples to (ii)  $\Rightarrow$  (i) are the strictly upper triangular matrices  $A$ , where  $|(I - A)^{-1}|$  and  $(I - |A|)^{-1}$  agree on and below the diagonal. The determination of all reducible  $A$  satisfying (i) or (ii) again seems to be a nontrivial problem.

**Theorem 3.** *Let  $A$  be an H-matrix. Then  $A$  and  $\langle A \rangle$  are nonsingular, and*

$$(9) \quad |A^{-1}| \leq \langle A \rangle^{-1}.$$

Moreover, if  $A$  is irreducible then one of the following holds:

- (i) (9) holds with strict (componentwise) inequality:  $|A^{-1}| < \langle A \rangle^{-1}$ .
- (ii) (9) holds with equality:  $|A^{-1}| = \langle A \rangle^{-1}$ , and  $A$  is diagonally equivalent to an M-matrix, i.e. nonsingular diagonal matrices  $D$  and  $D'$  exist such that  $D'AD$  is an M-matrix.

**Proof.** (9) is due to Ostrowski [3], but we reprove it in the process of obtaining the second part of the theorem. Write  $A = M - N$  where  $M = \text{Diag}(A_{ii})$ . Then  $\langle A \rangle = |M| - |N|$ . Since  $A$  is an H-matrix there is a real vector  $u > 0$  such that  $\langle A \rangle u > 0$ . Now  $\sigma(M^{-1}N) \leq \sigma(|M|^{-1}|N|) = \sigma(|M|^{-1}|N|) < 1$  by (6) and since

$|M|^{-1}|N|u = u - |M|^{-1}\langle A \rangle u < u$ . Hence  $A = M - N$  and  $\langle A \rangle = |M| - |N|$  are nonsingular, and by (6) and Theorem 2, we have

$$\begin{aligned} |A^{-1}| &= |(M - N)^{-1}| = |(I - M^{-1}N)^{-1}M^{-1}| = |(I - M^{-1}N)^{-1}||M|^{-1} \\ &\leq (I - |M^{-1}N|)^{-1}|M|^{-1} = (I - |M|^{-1}|N|)^{-1}|M|^{-1} \\ &= (|M| - |N|)^{-1} = \langle A \rangle^{-1}, \end{aligned}$$

so that (9) holds. Moreover, if  $|A^{-1}|$  and  $\langle A \rangle^{-1}$  have a common entry, so do  $|(I - M^{-1}N)^{-1}|$  and  $(I - |M^{-1}N|)^{-1}$ . Now if  $A$  is irreducible then  $M^{-1}N$  is irreducible, and by Theorem 2 there is a nonsingular diagonal matrix  $D$  such that  $C := D^{-1}M^{-1}ND$  is real and nonnegative. The vector  $v := |D|^{-1}u$  is positive and satisfies

$$Cv = |C|v = |D^{-1}M^{-1}ND||D|^{-1}u = |D|^{-1}|M|^{-1}|N|u < |D|^{-1}u = v.$$

Hence the matrix  $B := I - C$  satisfies  $\langle B \rangle = B$ ,  $\langle B \rangle v = v - Cv > 0$  and is therefore an  $M$ -matrix. But  $B = I - C = D^{-1}M^{-1}(M - N)D = D'AD$  with the nonsingular diagonal matrix  $D' = D^{-1}M^{-1}$ .

Conversely if  $D, D'$  are nonsingular diagonal matrices such that  $B := D'AD$  is an  $M$ -matrix then  $B^{-1} \geq 0$  so that  $|B^{-1}| = B^{-1} = \langle B \rangle^{-1}$  and

$$\begin{aligned} |A^{-1}| &= |DB^{-1}D'| = |D||B^{-1}||D'| = |D|\langle B \rangle^{-1}|D'| \\ &= (|D|^{-1}\langle B \rangle|D|)^{-1} = \langle A \rangle^{-1}. \quad \square \end{aligned}$$

#### References

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