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Some Tuple System Constructions with Applications to Resolvable Designs

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The doubling construction and the Kronecker product construction for Hadamard matrices are generalized to recursive constructions for homogeneous tuple systems. These generalizations lead in turn to recursive constructions for affine 2-designs and certain other resolvable designs.

INTRODUCTION

This paper is one of a number (e.g., Neumaier [13–15]) whose aim is to present various design-theoretical results within the framework of a uniform and transparent theory. The new theory uses the language of tuple systems, which enables many results (both old and new) to be stated and proved more briefly and elegantly. For example, affine 2-designs can be interpreted quite simply as tuple systems with certain homogeneity and maximality properties.

The key result of the paper is Theorem 3.2, which relates homogeneous tuple systems and resolvable 1-designs. Using this theorem it is a simple matter to translate the tuple system constructions derived in Section 2 into constructions for designs. ~~As~~ applications we obtain generalizations to affine designs of the doubling and Kronecker product constructions for Hadamard matrices (see Wallis *et. al.* [20]), simpler proofs of some results of Mavron [11] and Mullin [12], and some new constructions for affine designs of the types $AD(\mu m, m)$ and $SA(\mu m, m)$ (in the notation of Mavron [11]).

The paper concludes with a summary of the known facts about existence and non-existence of designs $AD(\mu m, m)$ and $SA(\mu m, m)$, and a selection of problems which deserve further study.

1. Homogeneous Tuple Systems

Let K and I be finite sets. An *I-tuple over K* is a map $x: I \to K$. As usual, we write $x = (x_i \mid i \in I)$ (or $x = (x_1, \ldots, x_n)$ if $I = \{1, 2, \ldots, n\}$). Elements of K are called *points*, and those of I *places*, and x_i is the *entry* of x in place i. A *tuple system* over K, I is a collection Q of (not necessarily distinct) I-tuples over K. For $I_0 \subseteq I$ and $x \in Q$ we can form $x|_{I_0} = (x_i \mid i \in I_0)$, and then $Q|_{I_0} = \{x|_{I_0} \mid x \in Q\}$ is the *restriction* of Q to I_0.

Let t, v, k, λ be positive integers with $v > 1$. A *$t-(v,k,\lambda)$-homogeneous tuple system* ($t-(v,k,\lambda)$-HTS for short) over K, I is a tuple system Q over K, I with $|K| = v$, $|I| = k$, and such that for all t-subsets $I_0 \subseteq I$ and all I_0-tuples z over K, there are exactly λ tuples $x \in Q$ with $x|_{I_0} = z$. Such an HTS is *resolvable* (RHTS for short) if it can be partitioned into $1-(v,k,1)$-HTS's $T_l (l \in L)$. Then $\tau = \{T_l \mid l \in L\}$ is a *resolution* of Q and the T_l are the *resolution classes* of τ. Clearly $|Q| = \lambda v^t$ and $|\tau| = \lambda v^{t-1}$. Two resolutions $\tau^{(1)} = \{T_l^{(1)} \mid l \in L\}$, $\tau^{(2)} = \{T_l^{(2)} \mid l \in L\}$ of Q are *orthogonal* if $|T_l^{(1)} \cap T_m^{(2)}| \leqslant 1$ for all $l, m \in L$.

The following example will be useful in the sequel.

1.1 Example. For $n \in \mathbb{Z}^+$ put $I_n = \{1, 2, \ldots, n\}$ and let Q be a $2-(v, k+1, 1)$-HTS over K and I_{k+1}. For $a \in K$, define $T_1(a) = \{x \in Q \mid x_{k+1} = a\}$ and $T_2(a) = \{x \in Q \mid x_k = a\}$. Then $Q|_{I_k}$ is a $2-(v,k,1)$-RHTS with resolution $\{T_1(a)|_{I_k} \mid a \in K\}$, and $Q|_{I_{k-1}}$ is a $2-(v, k-1, 1)$-RHTS with orthogonal resolutions $\{T_1(a)|_{I_{k-1}} \mid a \in K\}$ and $\{T_2(a)|_{I_{k-1}} \mid a \in K\}$.

We are mostly interested in HTS's with $t = 2$. For such HTS's we can find a sharp upper bound for k in terms of v and λ.

1.2 Theorem. *Let Q be a $2-(v,k,\lambda)$-HTS over K, I. Then $k \leqslant (\lambda v^2 - 1)/(\lambda - 1)$, with equality iff the following condition holds for some positive integer ρ:*

$$\textit{if } x, y \in Q,\ x \neq y, \textit{ then } x_i = y_i \textit{ for exactly } \rho \textit{ places } i \in I. \qquad (\mathrm{M}_\rho)$$

If (M_ρ) *is satisfied, then $\rho = (\lambda v - 1)/(v - 1)$ and $(\lambda - 1)/(v - 1)$ is an integer.*

The inequality in Theorem 1.2 is established in Bose and Bush [3] in terms of orthogonal arrays. (It is easily seen that a $t-(v,k,\lambda)$-HTS is

equivalent to an orthogonal array $OA(\lambda v^t, k, v, t)$.) In HTS language, a much simpler proof can be given. Fix $x \in Q$, and for $y \in Q$, $y \neq x$, define $\lambda_{xy} = \#\{i \in I \mid x_i = y_i\}$ and $s_n = \sum_{y \neq x} \lambda_{xy}^n$. Then s_0, s_1, s_2 are easily found, and the result follows from the inequality $s_2 s_0 \geqslant s_1^2$.

A $2-(v,k,\lambda)$-HTS with $k = (\lambda v^2 - 1)/(v - 1)$ will be called *maximal*.

Simple quadratic counting arguments also yield the following results, the first of which is a well-known theorem of Bose [1].

1.3 Theorem. *Let Q be a $1-(v,k,\lambda)$-HTS satisfying condition* (M_ρ). *Then $k(\lambda - 1) = \rho(\lambda v - 1)$. Also $k \geqslant \rho v + 1$, with equality iff Q is a maximal 2-HTS.*

1.4 Theorem. *Let Q be a $2-(v,k,\lambda)$-RHTS with resolution τ, and write $y \nsim x$ if x, y are in different classes of τ. Then $k \leqslant \lambda v$, with equality iff the following condition holds for some positive integer ρ:*

$$\textit{if } x, y \in Q,\ x \nsim y, \textit{ then } x_i = y_i \textit{ for exactly } \rho \textit{ places } i \in I. \qquad (\mathrm{MR}_\rho)$$

If (MR_ρ) *is satisfied, then $\rho = \lambda$.*

A $2-(v,k,\lambda)$-RHTS with $k = \lambda v$ will again be called *maximal*.

We now give two direct constructions for maximal HTS's. These examples were already known to Bose [2], but can be presented more clearly and compactly in HTS language.

1.5 Example. Let q be a prime power and $F_n = GF(q^n)$. Suppose $s \geqslant 2$ and $\alpha: F_s \to F_1$ is an F_1-linear epimorphism (for example, the trace map), and let I be a complete set of representatives of F_s^* (the multiplicative group of F_s) modulo F_1^*. Then $Q = \{(\alpha(ix) \mid i \in I) \mid x \in F_s\}$ is a maximal $2-(q, (q^s - 1)/(q - 1), q^{s-2})$-HTS over F, I.

1.6 Example. With the notation of the previous example, let $\beta: F_{m+n} \times F_{m+n} \to F_n$ be an F_1-bilinear map such that the equation $\beta(i, x) = a$ has a solution x for all $i \in F_{m+n}^*$, $a \in F_n$. (Such β exist for all $m \geqslant 0$, $n \geqslant 0$.) Then $Q = \{(a + \beta(i,b) \mid i \in F_{m+n}) \mid a \in F_n,\ b \in F_{m+n}\}$ is a maximal $2-(q^n, q^{m+n}, q^m)$-RHTS over F_n, F_{m+n}, with resolution $\{T(b) \mid b \in F_{m+n}\}$, where $T(b) = \{(a + \beta(i,b) \mid i \in F_{m+n}) \mid a \in F_n\}$.

2. Constructions for HTS

We give here some fairly general recursive constructions for HTS's. Most of the details of the proofs are omitted, as the use of HTS language makes them particularly simple. In Section 3 it will be shown how these results can be specialized to obtain design-theoretic constructions.

For convenience, we first state two results due (in other language) to Bose [2].

2.1 Lemma. *If there exists a* $2-(v, k, \lambda)$-RHTS, *then there exists a* $2-(v, k+1, \lambda)$-HTS.

2.2 Theorem. *Let Q be a $2-(v, k, \lambda)$-HTS over K, I, and Q' a $2-(v, k', \lambda v)$-RHTS over K, I' with resolution $\{T(x) \mid x \in Q\}$, where $I \cap I' = \phi$. Then the tuple system $Q^* = \{(x, y) \mid x \in Q, y \in T(x)\}$, where $(x, y)_i = x_i$ if $i \in I$, $(x, y)_i = y_i$ if $i \in I'$, is a $2-(v, k+k', \lambda v)$-HTS over K and $I \cup I'$. Moreover, if Q, Q' are both maximal, then so is Q^*.*

Our first construction uses a $2-(v, k_0, 1)$-HTS with two orthogonal resolutions. Example 1.1 gives a means of obtaining such an HTS.

2.3 Theorem. *Let Q be a $2-(v, k, \lambda)$-HTS over K, I, and Q_0 a $2-(v, k_0, 1)$-RHTS over K, I_0 with two orthogonal resolutions. Then a $2-(v, kk_0+1, \lambda v)$-RHTS, say, Q', can be constructed, and if Q is maximal and $k_0 = v-1$ then Q' is maximal.*

Proof. Suppose the resolutions of Q_0 are $\{T_i(a) \mid a \in K\}$, $i = 1, 2$. Define Q' over K, $(I \times I_0) \cup \{\infty\}$ by

$$Q' = \{(x, c) \mid x \in Q, c \in K\},$$

$$(x, c)_\infty = c,$$

$$(x, c)_{i,j} = z_j \qquad \text{iff} \quad T_1(c) \cap T_2(x_i) = \{z\},$$

for $i \in I$, $j \in I_0$.

A similar construction for 1-HTS's gives

2.4 Theorem. *Suppose Q is a $1-(v, k, \lambda)$-HTS satisfying condition (M_ρ) of Theorem 1.2, and Q_0 is a $2-(v, v, 1)$-RHTS. Then a $1-(v, kv+k-\rho v, \lambda v)$-HTS satisfying (M_k), say, Q^*, can be constructed.*

Proof. Suppose Q, Q_0 are over K, I and K, I_0, respectively, and that Q_0 has resolution $\{T(a) \mid a \in K\}$. Let I' be any $(k-\rho v)$-set disjoint from $I \times I_0$. (Note that $k - \rho v \geqslant 1$ by Theorem 1.3.) Define $Q^* = Q \times K$ over K, $(I \times I_0) \cup I'$ by

$$(x, c)_{i'} = c,$$

$$(x, c)_{i,j} = z_j \qquad \text{if} \quad z \in T(c), z_\infty = x_i,$$

for all $i' \in I'$, $i \in I$, $j \in I_0$, where ∞ is some fixed place in I_0.

The previous constructions required an RHTS with $\lambda = 1$. If RHTS's with $\lambda > 1$ are available, further constructions are possible.

2.5 Theorem. *Suppose that Q is a $2-(v, k, \lambda)$-HTS, Q' is a $2-(v, k', \lambda')$-HTS, and Q_0 is a $2-(v, k_0, \lambda' v)$-RHTS. Then a $2-(v, k_0 k + k', \lambda\lambda' v^2)$-HTS, say, Q^*, can be constructed. Moreover, if Q, Q', Q_0 are all maximal, then so is Q^*.*

Proof. Suppose that Q, Q', Q_0 all have point set K, and respective place sets I, I', I_0. Label the resolution classes of Q_0 as $T_z (z \in Q')$, and the tuples of T_z as $(z; a)$ $(a \in K)$. Define $Q^* = Q' \times Q$ over K, $(I_0 \times I) \cup I'$ as follows:

$$(z, x)_{i_0, i} = (z; x_i)_{i_0},$$

$$(z, x)_{i'} = z_{i'},$$

for all $z \in Q'$, $x \in Q$, $i \in I$, $i' \in I'$, $i_0 \in I_0$.

2.6 Theorem. *Suppose that Q is a $2-(v, k, \lambda)$-HTS and Q_0 is a 2-(v, k_0, λ_0)-RHTS. Then a $2-(v, k_0 k, \lambda_0 \lambda v)$-HTS, say, Q^*, can be constructed. Also, if Q is resolvable, then so is Q^*, and if Q, Q_0 are both maximal* RHTS's, *then so is Q^*.*

Proof. Suppose that Q, Q_0 are over K, I and K, I_0, respectively. Let S be any set with $|S| = \lambda_0 v$. Label the resolution classes of Q_0 as T_s $(s \in S)$, and the tuples of T_s as $(s; a)$ $(a \in K)$. Define $Q^* = S \times Q$ over K, $I_0 \times I$ by

$$(s, x)_{i_0, i} = (s; x_i)_{i_0},$$

for all $s \in S$, $x \in Q$, $i_0 \in I_0$, $i \in I$.

3. HTS's and Resolvable Designs

The following construction shows that resolvable 1-designs and 1-HTS's are equivalent concepts. Our notation and terminology for resolvable and affine designs is as in Mavron [11]. The dual notions of "parallelism" and "resolution" will be called "point parallelism" and "point resolution," respectively.

3.1 Construction. *Let Q be a $1-(v, k, \lambda)$-HTS over K, I. Define an incidence structure $\hat{Q}$ with point set Q, block set $I \times K$, incidence defined by $x \,\mathrm{I}\, (i, a)$ iff $x_i = a$, and parallelism defined by $(i, a) \| (j, b)$ iff $i = j$. Then $\hat{Q}$ is a resolvable $1-(\lambda v, \lambda, k)$-design. Conversely, starting with a resolvable 1-design, we can reverse the construction and generate a 1-HTS.*

3.2 THEOREM. *In Construction* 3.1,

(i) $\hat{Q}$ *is an affine* $1-(\lambda v^2, \lambda v, k)$*-design iff* Q *is a* $2\text{-}(v, k, \lambda)$-HTS,

(ii) $\hat{Q}$ *is a resolvable* $2-(\lambda v, \lambda, \rho)$*-design iff* Q *satisfies condition* (M_ρ),

(iii) $\hat{Q}$ *is an* $AD(\lambda v, v)$ *iff* Q *is a maximal* $2-(v, k, \lambda)$-HTS,

(iv) $\hat{Q}$ *is point resolvable iff* Q *is a* 1-RHTS,

(v) $\hat{Q}$ *is an* $SA(\lambda v, v)$ *iff* Q *is a maximal* $2-(v, k, \lambda)$-RHTS.

Proof. Parts (i) and (ii) are clear, and then (iii) is immediate from Theorem 1.2. For (iv), note that any resolution of Q is a point resolution of $\hat{Q}$, and conversely.

Now let Q be a maximal $2-(v, \lambda v, \lambda)$-RHTS. Then $\hat{Q}$ is a symmetric $1-(\lambda v^2, \lambda v, \lambda v)$-design, which is affine by (i) and point resolvable by (iv). Also, two points in different point parallel classes of $\hat{Q}$ are always on the same number of blocks because of condition (MR_λ) in Theorem 1.4, and hence $\hat{Q}$ is an $SA(\lambda v, v)$. The converse of (v) is similar.

Parts (i) and (iii) of Theorem 3.2 are due (in the language of orthogonal arrays) to Shrikhande and Bhagwandas [18]. As illustrations of Theorem 3.2, note that the HTS of Example 1.5 is essentially the $AD(q^{s-1}, q)$ of points and hyperplanes in $AG(s, q)$, while if $m = s-2$, $n = 1$ in Example 1.6, then $\hat{Q}$ is essentially the $SA(q^{s-1}, q)$ of points and hyperplanes in $AG(s, q)$, without those hyperplanes containing a fixed infinite point.

We now use Theorem 3.2 and our earlier HTS results to obtain information about affine 1-designs. To avoid clashes in notation, we will express the results in terms of design parameters rather than HTS parameters. In particular, m will denote the number of blocks in a parallel class, and μ the number of points common to two non-parallel blocks.

As a first application, Theorems 1.4 and 3.2 give the following result, which is due in part to Mavron [11].

3.3 COROLLARY. *Let* D *be an affine* $1-(\mu m^2, \mu m, r)$*-design. Then* $b \leqslant v$, *with equality iff* D *is an* $SA(\mu m, m)$.

The next result is also proved in Mavron [11].

3.4 THEOREM. *If there exists an* $AD(\mu m, m)$ *and an* $SA(\mu m^2, m)$, *then there exists an* $AD(\mu m^2, m)$.

Proof. Use Theorems 2.2 and 3.5.

Using RHTS's, we obtain

3.5 THEOREM. (i) *If there exist an* $SA(\mu m, m)$ *and an* $SA(\mu' m, m)$, *then there exists an* $SA(\mu\mu' m^2, m)$.

(ii) *If there exist an affine plane* π *of order* m *and either an* $AD(\mu m, m)$ *or an* $SA(\mu m, m)$, *then there exists an* $SA(\mu m^2, m)$.

Proof. (i) Use Theorems 2.6 and 3.2(v).

(ii) The plane π leads (via Example 1.1) to a $2-(m, m-1)$-HTS with two orthogonal resolutions. Given also an $AD(\mu m, m)$, the result follows by applying Theorem 2.3.

Finally, given π and an $SA(\mu m, m)$, delete one parallel class from π to obtain an $SA(m, m)$, and apply (i) with $\mu' = 1$.

The case of Theorem 3.5(ii) starting with an $AD(\mu m, m)$ is also proved in Mavron [11].

For affine 2-designs we have further constructions.

3.6 THEOREM. *Suppose there exists an affine plane* π *of order* m.

(i) *If there exists an* $AD(\mu m, m)$, *then there exists an* $AD(\mu m^2, m)$.

(ii) *If there exist an* $AD(\mu_1 m, m)$ *and an* $AD(\mu_2 m, m)$, *then there exists an* $AD(\mu_1\mu_2 m^3, m)$.

Proof. (i) Use Theorems 3.4 and 3.5(ii).

(ii) First apply Theorem 3.5(ii) to π and the $AD(\mu_1 m, m)$ to obtain an $SA(\mu_1 m^2, m)$. Then apply Theorem 2.5 to the tuple systems Q_0, Q' and Q corresponding, respectively, to the $SA(\mu_1 m^2, m)$, $AD(\mu_1 m, m)$ and $AD(\mu_2 m, m)$.

Part (i) of Theorem 3.6 is due to Kimberley [9], while special cases of part (ii) (e.g., μ a prime power and m a power of μ) are proved in Mavron [10]. Note that for $m = 2$, the results of Theorem 3.6 reduce to the doubling and Kronecker product constructions for Hadamard matrices (see [20]).

A straightforward application of Theorems 2.4 and 3.2(ii) gives

3.7 THEOREM. *If there exist an affine plane of order* m *and a resolvable* $2-(mk, k, \lambda)$*-design, then there exists a resolvable* $2-(m^2k, mk, \lambda(km-1)/(k-1))$*-design.*

Note that Theorem 3.7 is the same as Theorem 1 of Mullin [12], but without the assumption that the Latin squares be "patterned."

4. REMARKS AND PROBLEMS

We list some known results about existence and non-existence of designs $AD(\mu m, m)$ and $SA(\mu m, m)$, and pose some unanswered questions.

A1. (See [6].) An $AD(\mu m, m)$ can only exist if $m-1 \mid \mu-1$ and

(1) if m is odd, then $\mu = s^2$ or $\mu = m^* s^2$ for some odd s,

(2) if $m \equiv 2 \pmod 4$, then m^* has no prime divisor $\equiv 3 \pmod 4$, where m^* denotes the square-free part of m.

A2. The existence of an $AD(2\mu, 2)$ is equivalent to that of a Hadamard matrix of order 4μ, though the same Hadamard matrix can give rise to non-isomorphic designs. Such matrices are known to exist for all $\mu < 67$. Also, for each $n \geqslant 1$, there is an integer $s_0(n)$ such that an $AD(2^s n, 2)$ exists for all $s \geqslant s_0(n)$. See Wallis [19].

A3. Every known $AD(\mu m, m)$ with $m > 2$ has the parameters of an affine space, i.e., $m = q$, $\mu = q^s$ for some prime power q and some integer $s \geqslant 0$ (see Shrikhande [17]). Are there any $AD(\mu m, m)$ not having such parameters?

A4. Can any other methods (besides the Kronecker product) for constructing Hadamard matrices be generalized to the case $m > 2$? (See [20].) Can the methods of Wallis [19] be extended to $m > 2$ to prove results analogous to those in A2?

S1. In [16], Rajkundlia introduces the notion of a "Hadamard system" $H(n, d)$, which turns out to be equivalent to an $SA(nd, n)$. He shows that the existence of an $SA(2\mu, 2)$ is equivalent to that of a Hadamard matrix of order 2μ (so can only exist if $\mu = 1$ or is even). (This equivalence is also proved in Hine and Mavron [7].) In addition, he proves that an $SA(m(m-1), m)$ exists whenever m and $m-1$ are both prime powers (an example of an $SA(6, 3)$ also appears in [4]). He also derives a Kronecker product construction, which amounts to our Theorem 3.5(i). It now follows that whenever m and $m-1$ are prime powers, an $SA(m^j(m-1)^i, m)$ exists for all $i \geqslant 1$, $j \geqslant i$. Rajkundlia also constructs an isolated $SA(12, 3)$. In addition, he gives a construction for a symmetric $2-(m^3-m+1, m^2, m)$-design from an $SA(m-1, m-1)$ and an $SA(m(m-1), m)$, which enables him to establish the non-existence of $SA(30, 6)$, $SA(182, 14)$ and $SA(870, 30)$.

S2. Example 1.5 gives an $SA(\mu m, m)$ with $m = q^s$, $\mu = q^t$, for any prime power q and any integers $s > 0$, $t \geqslant 0$. Combined with the results of S1, this shows that an $SA(p^t m^j (m-1)^i, m)$ exists for all $t \geqslant 0$, $i \geqslant 1$, $j \geqslant i$, whenever m is a power of the prime p and $m-1$ is also a prime power. For $m = 3$, because of the existence of an $SA(12, 3)$, more can be said: an $SA(3^{j+1}2^i, 3)$ exists for all $j \geqslant 0$ and all $i = 0, 1, \ldots, 2j-2$.

S3. The existence of an $SA(2\mu, 2)$ for any prime power μ is established in Jungnickel [8], generalizing an earlier result of Butson [5].

S4. Are there any more $SA(\mu m, m)$ with m, μ not powers of the same prime? Are there any with m not a prime power?

S5. Are there non-existence conditions for $SA(\mu m, m)$ like those in A1 for $AD(\mu m, m)$ which include the results of S1?

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