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# Some sporadic geometries related to $PG(3, 2)$

By

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**Introduction.** The motivation for writing the present paper has been twofold. Firstly, in 1979, I discovered two sporadic geometries whose diagram (in the sense of Buekenhout [2]) is a Coxeter diagram. I wrote about them to Francis Buekenhout, and since he didn't answer I judged them to be of little relevance. But last year (1982) he asked me to publish my results since the local approach to buildings of Tits [18] generated much interest in such geometries ([15], [11], [12], [1]) which Kantor [11] now calls *geometries which are almost buildings* (GAB's).

Secondly, Arjeh Cohen and myself are working on a collection and survey of the known distance regular graphs. The present paper will provide a convenient reference for one such graph, a regular thin near octagon on 100 vertices (related to one of the GAB's), whose existence could be deduced before only indirectly from results of Calderbank and Wales [4] on the Hoffman-Singleton graph.

**1. The projective space $PG(3, 2)$.** We denote by $PG(n, q)$ the projective space of (projective) dimension $n$ over a finite field with $q$ elements (cf. McWilliams and Sloane [13]). The particular space $PG(3, 2)$ has 15 points, 35 lines and 15 planes. Each plane contains 7 points and 7 lines, each lines contains 3 points and is in 3 planes, and each point is in 7 lines and 7 planes. Any two lines in a plane intersect, and if two lines intersect they have a unique common point and lie in a unique plane. $PG(3, 2)$ has as automorphism group the group $L_4(2)$, known to be isomorphic to the alternating group $A_8$ on a set $X$ of 8 letters. This implies the existence of a description of the 35 lines of $PG(3, 2)$ as the $\frac{1}{2}\binom{8}{4}$ partitions of type $4^2$ (i.e. unordered partitions into two sets of size 4) of the 8 letters. Specifically we have

**Proposition 1.** *There is a bijection $l \leftrightarrow \bar{l}$ between the 35 lines $l$ of $PG(3, 2)$ and the 35 partitions $\bar{l}$ of type $4^2$ of an 8-set $X$, such that*

(i) *two lines $l_1$ and $l_2$ intersect if and only if the partitions $\bar{l}_1$ and $\bar{l}_2$ intersect in a partition of type $2^4$; the three lines determined by such a partition are concurrent and coplanar;*

(ii) *for any three distinct letters $\alpha, \beta, \gamma \in X$, and each point $a \in PG(3, 2)$, there is a unique line $l \ni a$ such that one of the two 4-sets of $\bar{l}$ contains $\alpha$, $\beta$, and $\gamma$.*

Proof. The bijection is easy to see inside the Steiner system $S = S(24, 8, 5)$. By Conway [6], whose terminology we use, the group of automorphisms of $S$ fixing an octad $X$ and a point $z \notin X$ acts as $A_8$ on $X$ and as $L_4(2)$ on $P = S - X - \{z\}$. With lines being the sets $P \cap Y$, where $Y$ is an octad intersecting $X$ in 4 points, $P$ becomes a projective space $PG(3, 2)$ and there is a one-to-one correspondence between the 35 partitions of type $4^2$ of $X$, the 35 sextetts defined by them, and the 35 lines induced on $P$ by such a sextett; cf. Bussemaker and Seidel [3]. Now a partition of type $2^4$ of $X$ defines three sextetts, corresponding to three concurrent and coplanar lines of $P$, and conversely, two intersecting lines of $P$ define two sextetts intersecting in a partition of type $2^{12}$ of $S$, hence they induce on $X$ a partition of type $2^4$. This proves (i).

To show (ii), let $\mathscr{D}_a$ be the set of 4-sets contained in some partition $\underline{l}$, where $l$ varies over the lines $\ni a$. Then $\mathscr{D}_a$ contains 14 quadruples, and by (i), any two of them have 0 or 2 common letters. Hence a triple $\alpha, \beta, \gamma$ of distinct letters is in at most one quadruple of $\mathscr{D}_a$. But the total number of triples in a quadruple of $\mathscr{D}_a$ is $14 \cdot 4 = 56 = \binom{8}{3}$ whence each triple occurs precisely once. □

In the terminology of designs (cf. McWilliams and Sloane [13]), the 15 sets $\mathscr{D}_a$, and dually the 15 sets $\mathscr{D}_\pi$ consisting of the 4-sets in the partition $\underline{l}$ for some line $l$ in a plane $\pi$ of $PG(3, 2)$, are $3-(8, 4, 1)$-designs. Any such design is isomorphic to the affine geometry $AG(3, 2)$ with automorphism group $2^3 L_3(2)$. Since $2^3 L_3(2)$ has index 15 in $A_8$ and 30 in $S_8$, there are precisely 30 such $3-(8, 4, 1)$-designs, falling into two orbits of 15 under $A_8$ — thus specifying the points and blocks of $PG(3, 2)$. We remark that this description can be used to give an elementary proof of Proposition 1; the Steiner system being used only for brevity.

As is apparent from the previous remark, the odd permutations of $S_8$ induce dualities of $PG(3, 2)$. In particular, the polarities of $PG(3, 2)$ are induced by odd involutions of $S_8$. An involution $(\alpha\beta)\ (\gamma\delta)\ (\varepsilon\zeta)$ fixes precisely the three lines defined by the partition $(\alpha\beta, \gamma\delta, \varepsilon\zeta, **)$ and the corresponding intersection point and embedding plane; hence it defines an orthogonal polarity. A transposition $(\alpha\beta)$ fixes the 15 lines corresponding to the partitions of shape $(\alpha\beta**, ****)$ and the 15 point-plane flags corresponding to the partitions of shape $(\alpha\beta, **, **, **)$. Hence each transposition defines a symplectic polarity, and absolute points and lines form a generalize quadrangle of order two.

We now fix a point $\infty \in X$, and write $Y = X \setminus \{\infty\}$. Then we may identify a triple (3-subset) $\alpha\beta\gamma$ of $Y$ with the partition $(\infty\alpha\beta\gamma, ****)$ of $X$. In this way we obtain from Proposition 1:

**Proposition 2.** *There is a bijection $l \leftrightarrow \underline{l}$ between the 35 lines $l$ of $PG(3, 2)$ and the 35 triples $\underline{l}$ of a 7-set $Y$ such that*

(i) *two lines $l_1$ and $l_2$ intersect if and only if the triples $\underline{l}_1$ and $\underline{l}_2$ have precisely one common point;*

(ii) *for any two distinct letters $\alpha, \beta \in Y$, and each point $a \in PG(3, 2)$, there is a unique line $l \ni a$ such that $\alpha, \beta \in \underline{l}$.* □

This second representation is invariant under $A_7$ only (the stabilizer of $\infty$). The points and planes may now be recovered as the 30 Fano planes $PG(2, 2)$ definable over $Y$, which again split into $15 + 15$ under $A_7$. The transpositions of $Y$ (i.e. those fixing $\infty$) again form symplectic polarities.

**2. Three linked partial 5-geometries.** A special feature of the representation of $PG(3, 2)$ as in Proposition 2 is that pairs of skew lines can be classified into two types, namely depending on whether the associated triples are disjoint or not. Moreover, this distinction is preserved by the symplectic polarities induced by transpositions of $Y$.

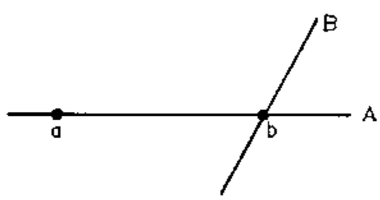
This fact has been used by Haemers [9] to construct the following graph $\Gamma$. Vertices (or *points*) of $\Gamma$ are the symbols $p_a$ and $p_l$, where $a$ is a point and $l$ is a line of $PG(3, 2)$; the neighbours of $p_a$ are the 7 vertices $p_l$ with $a \in l$, and the neighbours of $p_l$ are the 3 vertices $p_a$ with $a \in l$ and the 4 vertices $p_{l'}$ with $\underline{l} \cap l' = \emptyset$. It is immediate that $\Gamma$ is regular of valency $k = 7$, and since the number of vertices is $v = 15 + 35 = 50 = k^2 + 1$, it is the Hoffman-Singleton graph (Hoffman and Singleton [10]). The Hoffman-Singleton graph $\Gamma$ contains no triangles or quadrangles, and two nonadjacent points have a unique neighbour.

We now turn $\Gamma$ into a design $\mathscr{B}$ by defining $35 + 15 = 50$ blocks

$$B_l = \{p_a : a \in l\} \cup \{p_{l'} : |\underline{l} \cap l'| = 2\} \quad \text{for lines } l \text{ of } PG(3, 2),$$
$$B_\pi = \{p_a : a \notin \pi\} \cup \{p_l : l \in \pi\} \quad \text{for planes } \pi \text{ of } PG(3, 2).$$

The design $\mathscr{B}$ is selfdual since, for every symplectic polarity induced by a transposition of $Y$, the map $p_l \leftrightarrow B_{l\sigma}$, $p_a \leftrightarrow B_{a\sigma}$ is a polarity of $\mathscr{B}$. Since every block contains 15 points, $\mathscr{B}$ is a symmetric 1-design.

Now, in the terminology of Drake [8], a *partial $\Lambda$-geometry with block size $K$ and nexus $e$* is a symmetric 1-design $\mathscr{B}$ whose blocks contains $K$ points, such that two distinct points are in 0 or $\Lambda$ blocks, two distinct blocks are in 0 or $\Lambda$ points, and if $(a, B)$ is a nonincident point-block pair then there are $\Lambda e$ incident point-block pairs $(b, A)$ with $a \in A$, $b \in B$.




A parameter study (Neumaier [14]) shows that there are precisely two feasible parameter sets with $K = e + 3$, namely

$$K = 15, \quad \Lambda = 5, \quad e = 12 \quad \text{(with 50 points)},$$
$$K = 15, \quad \Lambda = 3, \quad e = 12 \quad \text{(with 85 points)}.$$

We show here that our design realizes the first possibility.

**Lemma.** *Two distinct points of $\Gamma$ are in 0 or 5 blocks of $\mathcal{B}$, depending on whether they are adjacent or not.*

Proof. Let $x, y$ be distinct nonadjacent points of $\Gamma$. Upto symmetry in $x$ and $y$, we have to consider four different situations. In each case we construct lines $m$ and/or planes $\pi$ defining 5 blocks $B_m$ and/or $B_\pi$.

(i) $x = p_a$, $y = p_b$ $(a \neq b)$. There are one line $m \ni a, b$ and four planes $\pi \not\ni a, b$.

(ii) $x = p_a$, $y = p_l$ $(a \notin l)$. Put $\underline{l} = \alpha\beta\gamma$. Then there are three lines $m \in a$ such that $\underline{m}$ intersects $\underline{l}$ in two points (Proposition 2 (ii)); also there are two planes $\pi \ni l$ with $a \notin \pi$.

(iii) $x = p_{l_1}$, $y = p_{l_2}$ $(|\underline{l}_1 \cap \underline{l}_2| = 1)$. Put $\underline{l}_1 = \alpha\beta\gamma$, $\underline{l}_2 = \alpha\delta\varepsilon$. Then there are four lines $m$ with $\underline{m} = \alpha\beta\delta, \alpha\beta\varepsilon, \alpha\gamma\delta, \alpha\gamma\varepsilon$; also there is a plane $\pi$ containing $l_1$ and $l_2$.

(iv) $x = p_{l_1}$, $y = p_{l_2}$ $(|\underline{l}_1 \cap \underline{l}_2| = 2)$. Put $\underline{l}_1 = \alpha\beta\gamma$, $\underline{l}_2 = \alpha\beta\delta$. Then there are three lines $m \neq l_1, l_2$ with $\alpha, \beta \in \underline{m}$, and two lines $m$ with $\underline{m} = \alpha\gamma\delta, \beta\gamma\delta$.

Therefore, two nonadjacent points are in at least 5 blocks. This gives us a total of $50 \cdot 42 \cdot 5 = 50 \cdot 15 \cdot 14$ triples $(x, y, B)$ with $x, y \in B$, $x \neq y$. Hence we have accounted for all such triples, and the assertion of the Lemma follows. □

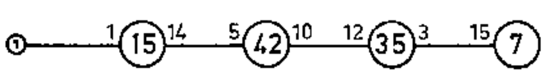
**Proposition 3.** *The design $\mathcal{B}$ is a partial 5-geometry with block size $K = 15$ and nexus $e = 12$.*

Proof. By the lemma, and the self-duality of $B$, only the condition on the nexus remains to be verified. Let $B$ be a block, and denote for $a \notin B$ by $t_a$ the number of points of $B$ adjacent with $a$. Then, since two nonadjacent points have a unique neighbour, we have (the sum extends over all $a \notin B$)

$$\sum 1 = 50 - 15\,, \quad \sum t_a = 15 \cdot 7\,,$$
$$\sum t_a(t_a - 1) = 15 \cdot 14\,, \quad \text{hence} \quad \sum (t_a - 3)^2 = 0\,.$$

Therefore, $t_a = 3$ for all $a \notin B$, and there are 12 points $b \in B$ nonadjacent with a fixed $a \notin B$. By the lemma, this yields $5 \cdot 15 = \Lambda \cdot e$ incident point-block pairs $(b, A)$ with $a \in A$, $b \in B$. □

By Drake [8], the incidence graph of a (proper) partial $\Lambda$-geometry is an imprimitive distance regular graph of diameter four, or, in the terminology of Shult and Yanushka [16], a thin regular near octagon. Its parameters are in the present case given by the following diagram.




In fact this graph $\Gamma^*$ is even distance transitive, with automorphism group $\mathrm{P}\Gamma\mathrm{U}_3(5)$. We show this by exhibiting the graph as a set of vectors inside the Leech lattice. We use the description of the Leech lattice $\Lambda_{24}$ given by Conway [6] and Curtis [7]. In accordance with our previous notation for the Steiner system $S(24, 8, 5)$, we arrange coordinates such that the 8 last coordinates form an octad $X$. We write $z$ for the first coordinate, $\infty$ for the first coordinate of $X$, $P$ for the complement of $X \cup \{z\}$, and $Y$ for $X - \{\infty\}$.

The set $\Sigma$ of vectors of $\Lambda_{24}$ of norm 80 orthogonal to the norm 48 vector $d = (5, 1^{23})$ consists of $2 \cdot 276$ vectors of shapes

$$\pm(3, 7(-1)^{22}) \quad (23 \text{ points} \neq z),$$
$$\pm(-1, 3^7(-1)^{16}) \quad (253 \text{ octads} \ni z).$$

They have mutual inner product $\pm 16$ and define 276 equiangular lines (cf. Taylor [17], Ex. 6.6, for the resulting regular two-graph). The three vectors

$$a = (4, 0^{15}, -4, 0^7),$$
$$b = (1, 1^{15}, 3, (-1)^7),$$
$$c = (0, 0^{15}, 2, 2^7)$$

lie in $\Lambda_{24}$; they have norm 32 and mutual inner product $-8$. Hence by Curtis [7], p. 565, the group of automorphisms of the Leech lattice fixing the set $\{\pm a, \pm b, \pm c\}$ is $G^* = 2 \cdot U_3(5) \cdot S_3$, where the subgroup $U_3(5)$ fixes each of $a$, $b$, $c$, $S_3$ permutes $a$, $b$, and $c$, and the center 2 is multiplication by $-1$. The conjugate subgroups $G_a$, $G_b$, $G_c$ of $G^*$ fixing $a$, $b$, or $c$, respectively have index six in $G^*$, hence are isomorphic to $U_3(5) \cdot 2 = \mathrm{P}\Gamma\mathrm{U}_3(5)$.

Since $a + b + c = d$, the group $G^*$ fixes the set $\Sigma$. In fact, $G^*$ has two orbits on $\Sigma$. One consists of $2 \cdot 126$ vectors of shape

$$\pm(-1, 3^5(-1)^{10}, -1, 3^2(-1)^5) \quad (126 \text{ octads} \ni z \text{ intersect } Y \text{ in two points})$$

orthogonal to $a$, $b$, $c$; they again define a regular twograph (cf. Taylor [16], Ex. 6.6). The other orbit consists of $6 \cdot 50$ vectors falling into 6 types depending on the value of the inner products $((x, a), (x, b), (x, c))$:

$$\text{type } \pm\text{I}: \quad (0, \pm 16, \mp 16),$$
$$\text{type } \pm\text{II}: \quad (\mp 16, 0, \pm 16),$$
$$\text{type } \pm\text{III}: \quad (\pm 16, \mp 16, 0).$$

The 50 type I vectors are of shapes

$$(-1, 3^7(-1)^8, -1, (-1)^7) \quad (15 \text{ octads} \ni z \text{ disjoint with } X),$$
$$(+1, (-3)^3 1^{12}, 1, 1^3(-3)^4) \quad (35 \text{ octads} \ni z, \infty \text{ intersect } X \text{ in 4 points}).$$

The 50 type II vectors are of shapes

$$(-3, (-7) 1^{14}, 1, 1^7) \quad (15 \text{ points} \in P),$$
$$(-1, 3^3(-1)^{12}, 3, 3^3(-1)^4) \quad (35 \text{ octads} \ni z, \infty \text{ intersect } X \text{ in 4 points}).$$

The 50 type III vectors are of shapes

$$(-3, 1^{15}, -7, 1^7) \quad (\infty),$$
$$(3, (-1)^{15}, -1, (-1)^6 7) \quad (7 \text{ points} \in Y),$$
$$(1, (-3)^5 1^{10}, -3, (-3) 1^6) \quad (42 \text{ octads} \ni z \text{ intersect } X \text{ in 2 points}).$$

The negatives of these vectors are of type $-$I, $-$II, and $-$III, respectively.

From this it is clear that the $A_7$ fixing $z$, $X$, and $\infty$ splits the type I and type II vectors as $15 + 35$, and the type III vectors as $1 + 7 + 42$. Indeed, the type III vector

$$e = (-3, 1^{15}, -7, 1^7)$$

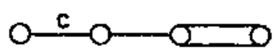
has inner product 16 with 15 type I vectors, 15 type II vectors, and 42 type III vectors, and inner product $-16$ with 35 type I vectors, 35 type II vectors, and 7 type III vectors. In particular, $G_a \cong \mathrm{P}\Gamma\mathrm{U}_3(5)$ acts as a rank 5 group on the 100 vectors of type II and III, with suborbits 15, 35, 1, 7, 42. Therefore, $G_e \cong \mathrm{P}\Gamma\mathrm{U}_3(5)$ also acts as a rank 5 group on the 100 vectors of type I and II, with the same suborbits. But the graph on these vectors obtained by joining a vector $x$ of type I with a vector $y$ of type II whenever $(x, y) = 16$ is easily seen to be isomorphic to the incidence graph $\Gamma^*$ of the partial 5-geometry $\mathscr{B}$ described above. Therefore, $\Gamma^*$ is distance transitive.

Remarks. 1. The vectors of type I, II, and III form a system of "linked" partial 5-geometries, related by the outer automorphism of $\mathrm{P}\Gamma\mathrm{U}_3(5)$ of order three. Thus the situation is similar to that of the linked partial $\Lambda$-geometries constructed by Cameron and Drake [5] from $D_4(q)$ with the triality automorphism.

2. Perhaps the neatest way of describing the graph $\Gamma^*$ is as follows: vertices of $\Gamma^*$ are the 100 cocliques of size 15 in the Hoffman-Singleton graph $\Gamma$; two cocliques are at distance 1, 2, 3, 4 if they intersect in 8, 5, 3, or 0 points, respectively. This can be deduced either from the above, or from a different representation of the Hoffman-Singleton graph described in Calderbank and Wales [4], Section 3.

**3. A sporadic geometry for $A_8$ and two GAB's.** Using the description of $PG(3, 2)$ in Section 1, we define a rank 4 geometry in the sense of Buekenhout [2]. (We assume the reader to be familiar with Buekenhout's language of diagrams and geometries.)

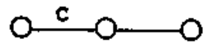
The 0-varieties of our geometry are the 8 *symbols* from $X$, and 1-varieties are the $\binom{8}{2} = 28$ *transpositions* on $X$ (equivalently, symplectic polarities of $P$). As 2-varieties we take the 35 *lines* of $P$, and as 3-varieties the 15 *points* of $P$. We define incidence as follows. A symbol $\alpha$ is incident with the transpositions moving $\alpha$, with all lines, and with all points. A transposition $(\alpha\beta)$ is incident with the two symbols $\alpha, \beta$, with the 15 lines fixed by $(\alpha\beta)$, and with all points. A line $l$ is incident with all symbols, with the transpositions fixing $l$, and with the points on $l$. Finally, a point $x$ is incident with all symbols, all transpositions, and the lines which contain $x$. This defines a strongly connected geometry with automorphism group $A_8$ and Buekenhout diagram

$A_8$ 


The verification of the axioms and the diagram is a straightforward consequence of the results of Section 1. The intersection property does not hold since, e.g., every transposition is incident with every point.

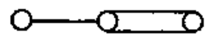
Two residues of this geometry (both without the intersection property) are also interesting, namely the residue of a point, with diagram

$2^3 L_3(2)$




and the residue of a symbol, with diagram

$A_7$



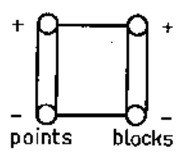
This latter geometry is particularly interesting since it has as diagram a Coxeter diagram; thus, in the terminology of Kantor [11] it is a GAB (geometry which is almost a building). It has been discovered independently by Aschbacher and and Smith [1].

A second GAB arises from the partial 5-geometry $\mathscr{B}$ discussed in Section 2. Indeed, take as varieties the symbols $a^+$, $a^-$, $B^+$, $B^-$, where $a$ is a point and $B$ is a block of $\mathscr{B}$. Incidence is defined by

$$a^\delta I b^\varepsilon \quad \text{if } a, b \text{ nonadjacent}, \quad \delta \neq \varepsilon,$$
$$a^\delta I B^\varepsilon \quad \text{if } \quad a \in B, \quad \delta = \varepsilon \quad \text{or} \quad a \notin B, \quad \delta \neq \varepsilon,$$
$$A^\delta I B^\varepsilon \quad \text{if } \quad A \cap B = \emptyset, \quad \delta \neq \varepsilon.$$

From the results of Section 2 it is straightforward to show that the geometry has the diagram

$U_3(5)$




with rank 3 residues isomorphic to the $A_7$-GAB. This geometry is also described in Kantor [12] in purely group-theoretic terms.

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