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INEQUALITIES FOR POINT STABLE DESIGNS

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ABSTRACT

A design with incidence matrix A is called point stable if $AA^TJ = \alpha J$ for some integer α, where J is the all-one-matrix. We derive inequalities for point stable designs generalizing the MULLIN-VANSTONE-inequality for (r,λ)-designs, the FISHER-inequality for 2-designs, the HANANI-inequality for transversal designs, an inequality for partial geometric designs and the CONNOR-MAJUMDAR-SHAH-AGRAWAL-inequalities for the intersection numbers of 1-designs.

1. NOTATIONS

A design (or incidence structure) $D = (P,B,I)$ with $(v\times b)$-incidence matrix A ($v = |P|$, $b = |B|$) is called *point stable* if $NJ = JN$, where $N = AA^T$ and J is the all-one-matrix. D is point stable iff $NJ = \alpha J$ for some integer α. Then D is called a *PSI(α)*, and a *PSI(α,r)* if furthermore $AJ = rJ$. A design D is a PSI(α) iff $\sum_{pIB} [B] = \alpha$ for all points p, where [B] denotes the number of points on the block B and α is the maximal eigenvalue of N and its multiplicity equals the number of connected components of D.

EXAMPLES: Any 1-(v,k,r)-design is a PSI(rk), any (r,λ)-design is a PSI($r+\lambda(v-1)$).

In the following we always require the nondegeneracy conditions

(1.1) $2 \leqslant [p] < b$, $2 \leqslant [B] < v$ for all points p and blocks B, where [p] is the number of blocks through p. Then $\alpha < \sigma$, where $\sigma = |I| = \operatorname{trace} N$ (cf. [12]).

2. SOME BASIC INEQUALITIES

First we mention two variance inequalities.

THEOREM 2.1 [12]: *Let* D *be a* PSI(α). *Then*

$$\alpha \geq \frac{\sigma^2}{vb},$$

with equality iff $[B] = \frac{\sigma}{b}$ *for all blocks* B.

PROOF: $0 \leq \sum_{B \in \mathcal{B}} ([B] - \frac{\sigma}{b})^2 = \sum_{B \in \mathcal{B}} [B]^2 - \frac{\sigma^2}{b} = v\alpha - \frac{\sigma^2}{b}$.

REMARK: If D is a 1-(v,k,r)-design, then Thm.2.1. yields the first FISHER-equation vr = bk.

Since an (r,λ)-design is a PSI($r+\lambda(v-1)$) with $\sigma = vr$ we obtain

COROLLARY 2.2 (MULLIN-VANSTONE [9]): *Let* D *be an* (r,λ)*-design. Then*

$$b \geq \frac{vr^2}{r+\lambda(v-1)},$$

with equality iff D *is a* 2*-design.*

REMARK: McCARTHY, VANSTONE [8] obtained this inequality from a determinantal condition on CONNOR's characteristic matrix.

For any point p let $w(p) = \sum_{q \in \mathcal{P} \setminus \{p\}} [p,q]^2$, i.e. the number of paths (p,B,q,C,p) with $q \neq p$.

THEOREM 2.3: *Let* D *be a* PSI(α). *Then*

$$w(p) \geq (v-1)^{-1}(\alpha - [p])^2 \quad \textit{for all points } p,$$

with equality iff $[p,q] = (v-1)^{-1}(\alpha - [p])$ *for all* $q \neq p$.

PROOF: $0 \leq \sum_{q \in \mathcal{P} \setminus \{p\}} ([p,q] - (v-1)^{-1}(\alpha - [p]))^2 = w(p) - (v-1)^{-1}(\alpha - [p])^2$

COROLLARY 2.4: *Let* D *be a* PSI(α,r). *Then*

$$w(p) \geq (v-1)^{-1}(\alpha - r)^2 \text{ for all points } p,$$

with equality iff D *is an* (r,λ)*-design with* $\lambda = (v-1)^{-1}(\alpha - r)$.

REMARK: Note that $\lambda = (v-1)^{-1}(\alpha - r)$ yields the second FISHER-equation $\lambda(v-1) = r(k-1)$ for 2-(v,k,λ)-designs.

3. AN EIGENVALUE INEQUALITY

The following inequality generalizes inequalities for semi partial geometric designs [12] and partial geometric designs [10].

THEOREM 3.1: *Let* D *be a connected* PSI(α), $\operatorname{spec}N \setminus \{0\} = \{\alpha, \rho_1, \ldots, \rho_m\}$, $\alpha > \rho_1 > \ldots > \rho_m$ *and let the corresponding multiplicities be* $1, \tau_1, \ldots, \tau_m$. *Then*

$$\rho_m \geq \frac{\sigma - \alpha - \rho_1\tau_1 - \ldots - \rho_{m-1}\tau_{m-1}}{v - 1 - \tau_1 - \ldots - \tau_{m-1}}$$

with equality iff $\det N > 0$.

PROOF: Clearly $\sigma = \alpha + \sum_{i=1}^{m} \rho_i\tau_i$ and $1 + \sum_{i=1}^{m} \tau_i \leq v$ with equality iff $0 \notin \operatorname{spec}N$.

COROLLARY 3.2 [12]: *Let* D *be a connected* PSI(α) *with* $\operatorname{spec}N \setminus \{0\} = \{\alpha, \rho\}$, $\alpha > \rho$, *then*

$$\rho \geq \frac{\sigma - \alpha}{v - 1},$$

with equality iff D *is an* (r,λ)*-design.*

PROOF: A connected point stable design is an (r,λ)-design iff $|\operatorname{spec}N| = 2$ (cf. [12]).

A *partial geometric design* is a 1-design satisfying $NA - \rho A = tJ$ for some $\rho, t \in \mathbb{R}$ (in fact ρ, t are positive integers). BOSE, BRIDGES, SHRIKHANDE [3] proved that a connected 1-design is partial geometric iff $|\operatorname{spec}N \setminus \{0\}| = 2$.

Now application of Corollary 3.2 and some elementary calculations show

COROLLARY 3.3 [10]: *Let* D *be a partial geometric* 1-(v,k,r)-*design and* $NA - \rho A = tJ$, *then*

$$t \geq k(r - \rho),$$

with equality iff D *is a 2-design.*

REMARK: Since the dual of a partial geometric design is partial geometric (with the same ρ, t), also the dual inequality $t \geq r(k - \rho)$ holds, with equality iff the dual of D is a 2-design. Therefrom one may derive (cf. [10])

(i) the FISHER-inequality $b \geq v$ for 2-(v,k,λ)-designs, with equality iff D is a symmetric 2-design,

(ii) the HANANI-inequality [6] $k \leq (\lambda u^2 - 1)(u - 1)^{-1}$ for transversal designs TD[k,λ,u] ($u > 1$), with equality iff the dual of D is a 2-design (and hence an affine design).

4. DETERMINANT FORMULAE

Let D be a design and m a nonnegative integer.

4.1 The m+2 ($v \times v$)-matrices $N^0, N^1, N^2, \ldots, N^m, J$ are dependent (in $\mathbb{R}^{v \cdot v}$) iff there is a real monic polynomial f(X) of degree $\partial f(X) \leq m$ such that $f(N) = tJ$ for some real t.

4.2 The m+2 ($v \times b$)-matrices $A, NA, N^2A, \ldots, N^mA, J$ are dependent (in $\mathbb{R}^{v \cdot v}$) iff there is a real monic polynomial f(X) of degree $\partial f(X) \leq m$ such that $f(N)A = tJ$ for some real t.

DEFINITION 4.3 [13]: The *point rank Pr(D)* (resp. the *rank R(D)*) is the minimal degree of a real monic polynomial f(X) such that $f(N) = tJ$ (resp. $f(N)A = tJ$) for some real t.

EXAMPLES: (i) Pr(D) = 1 iff D is an (r,λ)-design.

(ii) Supposed that D is point stable. Then R(D) = 1 iff D is partial geometric (cf. [13]). From (4.1.,4.2.) it is easy to see that

(1) $N^0, N^1, \ldots, N^m$, J are dependent iff $Pr(D) \leq m$,

(2) $A, NA, \ldots, N^mA$, J are dependent iff $R(D) \leq m$.

Now let D be a PSI(α) and t_s be the trace of N^s (s a nonnegative integer), hence

$t_s = \sum_{p\in P} (N^s)_{p,p}$ is the number of closed paths of length s. Then the Gram-matrices

$G_m = \mathrm{Gram}(N^0,N^1,\dots,N^m,J) = (g_{ij})_{i,j=0,\dots,m+1}$ and

$H_m = \mathrm{Gram}(A,NA,\dots,N^mA,J) = (h_{ij})_{i,j=0,\dots,m+1}$

of the vectors $N^0,\dots,N^m$, $J \in \mathbb{R}^{vv}$ (resp. $A,NA,\dots,N^mA,J \in \mathbb{R}^{vb}$) have the following entries: For $0 \leq i,j \leq m$

$$g_{ij} = \sum_{p,q\in P} (N^i)_{p,q}(N^j)_{p,q} = \sum_{p\in P} (N^{i+j})_{p,p} = t_{i+j} ,$$

$$g_{i,m+1} = g_{m+1,i} = \sum_{p,q\in P} (N^i)_{p,q} = J_{1,v}N^iJ_{v,1} = \alpha^i v,$$

$$g_{m+1,m+1} = v^2$$

$$h_{ij} = \sum_{p\in P\ B\in B} (N^iA)_{p,B}(N^jA)_{p,B} = \sum_{p\in P} (N^iA\ (N^jA)^T)_{p,p} = t_{i+j+1} ,$$

$$h_{i,m+1} = h_{m+1,i} = \sum_{p,B} (N^iA)_{p,B} = J_{1,v}\ N^iA\ J_{b,1} = \alpha^i\sigma ,$$

$$h_{m+1,m+1} = vb.$$

Using that G_m and H_m are Gram-matrices we obtain from (1) and (2) the following inequalities.

(3) $\det G_m \geq 0$, with equality iff $\Pr(D) \leq m$,

(4) $\det H_m \geq 0$, with equality iff $R(D) \leq m$.

In (3) and (4) we may replace G_m and H_m by

(5) $\bar{G}_m = (t_{i+j} - \alpha^{i+j})_{i,j=0,\dots,m}$ and

(6) $\bar{H}_m = (t_{i+j+1} - \alpha^{i+j}\frac{\sigma^2}{vb})_{i,j=0,\dots,m}$ respectively, since

(7) $\det G_m = v^2 \det \bar{G}_m$,

(8) $\det H_m = vb \det \bar{H}_m$,

which is easily seen using the transformations

$\bar{g}_{ij} = g_{ij} - \alpha^i v^{-1} g_{m+1,j}$ and

$\bar{h}_{ij} = h_{ij} - \frac{\alpha^i \sigma}{vb} h_{m+1,j}$, where $0 \leq i \leq m$, $0 \leq j \leq m+1$.

Hence we obtain from (3,4,7,8)

THEOREM 4.6: *Let* D *be a* PSI(α), m *a nonnegative integer. Then*

(9) $\det \bar{G}_m \geq 0$, *with equality iff* $Pr(D) \leq m$,

(10) $\det \bar{H}_m \geq 0$, *with equality iff* $R(D) \leq m$.

As an example we discuss inequality (9) for $m = 1$. Then we get

COROLLARY 4.7: *Let* D *be a* PSI(α). *Then*

$$t_2 \geq (\sigma - \alpha)^2 (v-1)^{-1} + \alpha^2 ,$$

with equality iff D *is an* (r,λ)*-design.*

PROOF: $\bar{G}_1 = \begin{pmatrix} v-1 & \sigma-\alpha \\ \sigma-\alpha & t_2-\alpha^2 \end{pmatrix}$ and by (1.1.) and the example in 4.3 we obtain

$Pr(D) \leq 1$ iff $Pr(D)=1$ iff D is an (r,λ)-design.

Using the number c_2 of proper two-gons (p,B,q,C,p) $p \neq q$, $B \neq C$ we get

COROLLARY 4.8: *Let* D *be a* PSI(α,r). *Then*

$$c_2 \geq (v-1)^{-1}\, v\, (\alpha - r)\, (\alpha - r - v + 1) ,$$

with equality iff D *is an* (r,λ)*-design.*

From Corollary 4.8 we may derive again the FISHER-inequality for 2-designs, the inequality 3.2 for partial geometric designs and the inequality for partial 1-designs D : $v-1 \geq r(k-1)$, with equality iff D is a 2-$(v,k,1)$-design.

The following theorem sharpens corollary 4.7.

THEOREM 4.9: *Let* D *be a* PSI(α). *Then*

$$t_2 \geq (v-1)^{-1}\, (v\alpha^2 - 2\alpha\sigma + v \sum_{p \in P} [p]^2) \geq (\sigma - \alpha)^2 (v-1)^{-1} + \alpha^2,$$

the first equality holds iff D *is an* (r,λ)*-design,* *the second equality holds iff* D *is a* PSI(α,r).

PROOF: From $t_2 = \sum_{p\in P} (w(p) + [p]^2)$ and Thm. 2.3. we obtain the first inequality. The second inequality is equivalent to the variance-inequality (compare Thm. 2.1.) $\sum_{p\in P} [p]^2 \geq \sigma^2 v^{-1}$ with equality iff $[p] = \sigma v^{-1}$ for all points p.

5. BOUNDS FOR THE CONNECTION NUMBERS OF REGULAR POINT STABLE DESIGNS

The following theorem generalizes and improves the inequalities of AGRAWAL [1,2] for the intersection numbers $[B,C]$ of different blocks B,C of a 1-design. The AGRAWAL-inequalities contain the SHAH-inequalities for PBIBD's [11], and the CONNOR-MAJUMDAR-inequalities for 2-designs [5,7].

THEOREM 5.1: *Let* D *be a connected* PSI(α,r),

$$\text{specN} = \{\alpha, \rho_1,\ldots, \rho_m\}\ ,\ \alpha > \rho_1 > \ldots > \rho_m.$$

Then for any two different points p,q *we have*

$$\max\,\{2r - b,\ r - \rho_1,\ 2(\alpha - \rho_m)v^{-1} + \rho_m - r\} \leq [p,q] \leq$$
$$\min\,\{r - \rho_m,\ 2(\alpha - \rho_1)v^{-1} + \rho_1 - r\}.$$

The proof will be given elsewhere.

COROLLARY 5.2: (AGRAWAL [1,2]): *Let* D *be a connected* 1-(v,k,r)-*design*,

$$\text{specN} = \{rk, \rho_1,\ldots, \rho_m\}\ ,\ rk > \rho_1 > \ldots > \rho_m.$$

Then for any two different blocks B,C *we have*

$$\max\,\{2k - v,\ k - \rho_1,\ 2rkb^{-1} - k\} \leq [B,C] \leq$$
$$\min\,\{k,\ 2(rk - \rho_1)b^{-1} + \rho_1 - k\}.$$

REMARK: It is easily seen that the application of Thm. 5.1. to the dual of the 1-design D in Cor. 5.2. yields exactly the AGRAWAL-bounds if $\det(A^TA) = 0$, where A is the incidence matrix of D, while otherwise both bounds $2rkb^{-1} - k \leq [B,C] \leq k$ are improved by Thm. 5.1.

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