

INEQUALITIES FOR POINT STABLE DESIGNS

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ABSTRACT

A design with incidence matrix A is called point stable if $AA^T J = \alpha J$ for some integer α , where J is the all-one-matrix. We derive inequalities for point stable designs generalizing the MULLIN-VANSTONE-inequality for (r, λ) -designs, the FISHER-inequality for 2-designs, the HANANI-inequality for transversal designs, an inequality for partial geometric designs and the CONNOR-MAJUMDAR-SHAH-AGRAWAL-inequalities for the intersection numbers of 1-designs.

1. NOTATIONS

A design (or incidence structure) $D = (P, B, I)$ with $(v \times b)$ -incidence matrix A ($v = |P|$, $b = |B|$) is called *point stable* if $NJ = JN$, where $N = AA^T$ and J is the all-one-matrix. D is point stable iff $NJ = \alpha J$ for some integer α . Then D is called a $PSI(\alpha)$, and a $PSI(\alpha, r)$ if furthermore $AJ = rJ$. A design D is a $PSI(\alpha)$ iff $\sum_{p \in B} [B] = \alpha$ for all points p , where $[B]$ denotes the number of points on the block B and α is the maximal eigenvalue of N and its multiplicity equals the number of connected components of D .

EXAMPLES: Any $1-(v, k, r)$ -design is a $PSI(rk)$, any (r, λ) -design is a $PSI(r + \lambda(v-1))$.

In the following we always require the nondegeneracy conditions (1.1) $2 \leq [p] < b$, $2 \leq [B] < v$ for all points p and blocks B , where $[p]$ is the number of blocks through p . Then $\alpha < \sigma$, where $\sigma = |I| = \text{trace } N$ (cf. [12]).

2. SOME BASIC INEQUALITIES

First we mention two variance inequalities.

THEOREM 2.1 [12]: *Let D be a PSI(α). Then*

$$\alpha \geq \frac{\sigma^2}{vb},$$

with equality iff $[B] = \frac{\sigma}{b}$ for all blocks B.

PROOF: $0 \leq \sum_{B \in \mathcal{B}} ([B] - \frac{\sigma}{b})^2 = \sum_{B \in \mathcal{B}} [B]^2 - \frac{\sigma^2}{b} = v\alpha - \frac{\sigma^2}{b}.$

REMARK: If D is a $1-(v,k,r)$ -design, then Thm.2.1. yields the first FISHER-equation $vr = bk.$

Since an (r,λ) -design is a $\text{PSI}(r+\lambda(v-1))$ with $\sigma = vr$ we obtain

COROLLARY 2.2 (MULLIN-VANSTONE [9]): *Let D be an (r,λ) -design. Then*

$$b \geq \frac{vr^2}{r+\lambda(v-1)},$$

with equality iff D is a 2-design.

REMARK: MCCARTHY, VANSTONE [8] obtained this inequality from a determinantal condition on CONNOR's characteristic matrix.

For any point p let $w(p) = \sum_{q \in \mathcal{P}(p)} [p,q]^2$, i.e. the number of paths (p,B,q,C,p) with $q \neq p$.

THEOREM 2.3: *Let D be a PSI(α). Then*

$$w(p) \geq (v-1)^{-1}(\alpha - [p])^2 \text{ for all points } p,$$

with equality iff $[p,q] = (v-1)^{-1}(\alpha - [p])$ for all $q \neq p$.

PROOF: $0 \leq \sum_{q \in \mathcal{P}(p)} ([p,q] - (v-1)^{-1}(\alpha - [p]))^2 = w(p) - (v-1)^{-1}(\alpha - [p])^2$

COROLLARY 2.4: *Let D be a PSI(α,r). Then*

$$w(p) \geq (v-1)^{-1}(\alpha - r)^2 \text{ for all points } p,$$

with equality iff D is an (r, λ) -design with $\lambda = (v-1)^{-1}(\alpha - r)$.

REMARK: Note that $\lambda = (v-1)^{-1}(\alpha - r)$ yields the second FISHER-equation

$$\lambda(v-1) = r(k-1) \text{ for } 2-(v, k, \lambda)\text{-designs.}$$

3. AN EIGENVALUE INEQUALITY

The following inequality generalizes inequalities for semi partial geometric designs [12] and partial geometric designs [10].

THEOREM 3.1: Let D be a connected $PSI(\alpha)$, $\text{spec}N \setminus \{0\} = \{\alpha, \rho_1, \dots, \rho_m\}$, $\alpha > \rho_1 > \dots > \rho_m$ and let the corresponding multiplicities be $1, \tau_1, \dots, \tau_m$. Then

$$\rho_m \geq \frac{\alpha - \alpha - \rho_1 \tau_1 - \dots - \rho_{m-1} \tau_{m-1}}{v - 1 - \tau_1 - \dots - \tau_{m-1}}$$

with equality iff $\det N > 0$.

PROOF: Clearly $\sigma = \alpha + \sum_{i=1}^m \rho_i \tau_i$ and $1 + \sum_{i=1}^m \tau_i \leq v$

with equality iff $0 \notin \text{spec}N$.

COROLLARY 3.2 [12]: Let D be a connected $PSI(\alpha)$ with $\text{spec}N \setminus \{0\} = \{\alpha, \rho\}$, $\alpha > \rho$, then

$$\rho \geq \frac{\sigma - \alpha}{v - 1},$$

with equality iff D is an (r, λ) -design.

PROOF: A connected point stable design is an (r, λ) -design iff $|\text{spec}N| = 2$ (cf. [12]).

A partial geometric design is a 1-design satisfying $NA - \rho A = tJ$ for some $\rho, t \in \mathbb{R}$ (in fact ρ, t are positive integers). BOSE, BRIDGES, SHRIKHANDE [3] proved that a connected 1-design is partial geometric iff $|\text{spec}N \setminus \{0\}| = 2$.

Now application of Corollary 3.2 and some elementary calculations show

COROLLARY 3.3 [10]: Let D be a partial geometric 1 -(v, k, r)-design and $NA - \rho A = tJ$, then

$$t \geq k(r - \rho),$$

with equality iff D is a 2-design.

REMARK: Since the dual of a partial geometric design is partial geometric (with the same ρ, t), also the dual inequality $t \geq r(k - \rho)$ holds, with equality iff the dual of D is a 2-design. Therefrom one may derive (cf. [10])

- (i) the FISHER-inequality $b \geq v$ for 2-(v, k, λ)-designs, with equality iff D is a symmetric 2-design.
- (ii) the HANANI-inequality [6] $k \leq (\lambda u^2 - 1)(u - 1)^{-1}$ for transversal designs $TD[k, \lambda, u]$ ($u > 1$), with equality iff the dual of D is a 2-design (and hence an affine design).

4. DETERMINANT FORMULAE

Let D be a design and m a nonnegative integer.

4.1 The $m+2$ ($v \times v$)-matrices $N^0, N^1, N^2, \dots, N^m, J$ are dependent (in $\mathbb{R}^{v \times v}$) iff there is a real monic polynomial $f(X)$ of degree $\deg f(X) \leq m$ such that $f(N) = tJ$ for some real t .

4.2 The $m+2$ ($v \times b$)-matrices $A, NA, N^2A, \dots, N^mA, J$ are dependent (in $\mathbb{R}^{v \times v}$) iff there is a real monic polynomial $f(X)$ of degree $\deg f(X) \leq m$ such that $f(N)A = tJ$ for some real t .

DEFINITION 4.3 [13]: The *point rank* $Pr(D)$ (resp. the *rank* $R(D)$) is the minimal degree of a real monic polynomial $f(X)$ such that $f(N) = tJ$ (resp. $f(N)A = tJ$) for some real t .

EXAMPLES: (i) $Pr(D) = 1$ iff D is an (r, λ) -design.

(ii) Supposed that D is point stable. Then $R(D) = 1$ iff D is partial geometric (cf. [13]). From (4.1., 4.2.) it is easy to see that

(1) N^0, N^1, \dots, N^m, J are dependent iff $Pr(D) \leq m$,

(2) A, NA, \dots, N^mA, J are dependent iff $R(D) \leq m$.

Now let D be a $PSI(\alpha)$ and t_s be the trace of N^s (s a nonnegative integer),

hence

$t_s = \sum_{p \in P} (N^s)_{p,p}$ is the number of closed paths of length s . Then the Gram-matrices

$$G_m = \text{Gram}(N^0, N^1, \dots, N^m, J) = (g_{ij})_{i,j=0, \dots, m+1} \quad \text{and}$$

$$H_m = \text{Gram}(A, NA, \dots, N^m A, J) = (h_{ij})_{i,j=0, \dots, m+1}$$

of the vectors $N^0, \dots, N^m, J \in \mathbb{R}^{V \times V}$ (resp. $A, NA, \dots, N^m A, J \in \mathbb{R}^{V \times b}$) have the following entries: For $0 \leq i, j \leq m$

$$g_{ij} = \sum_{p, q \in P} (N^i)_{p,q} (N^j)_{p,q} = \sum_{p \in P} (N^{i+j})_{p,p} = t_{i+j},$$

$$g_{i, m+1} = g_{m+1, i} = \sum_{p, q \in P} (N^i)_{p,q} = J_{i,v} N^i J_{v,1} = \alpha^i v,$$

$$g_{m+1, m+1} = v^2$$

$$h_{ij} = \sum_{p \in P} \sum_{B \in B} (N^i A)_{p,B} (N^j A)_{p,B} = \sum_{p \in P} (N^i A (N^j A)^T)_{p,p} = t_{i+j+1},$$

$$h_{i, m+1} = h_{m+1, i} = \sum_{p, B} (N^i A)_{p,B} = J_{i,v} N^i A J_{b,1} = \alpha^i \sigma,$$

$$h_{m+1, m+1} = vb.$$

Using that G_m and H_m are Gram-matrices we obtain from (1) and (2) the following inequalities.

(3) $\det G_m \geq 0$, with equality iff $\text{Pr}(D) \leq m$,

(4) $\det H_m \geq 0$, with equality iff $\text{R}(D) \leq m$.

In (3) and (4) we may replace G_m and H_m by

(5) $\bar{G}_m = (t_{i+j} - \alpha^{i+j})_{i,j=0, \dots, m}$ and

(6) $\bar{H}_m = (t_{i+j+1} - \alpha^{i+j} \frac{\sigma^2}{vb})_{i,j=0, \dots, m}$ respectively, since

(7) $\det G_m = v^2 \det \bar{G}_m$,

(8) $\det H_m = vb \det \bar{H}_m$,

which is easily seen using the transformations

$$\bar{g}_{ij} = g_{ij} - \alpha^i v^{-1} g_{m+1, j} \quad \text{and}$$

$$\bar{h}_{ij} = h_{ij} - \frac{\alpha^i}{v\beta} h_{m+1,j}, \quad \text{where } 0 \leq i \leq m, 0 \leq j \leq m+1.$$

Hence we obtain from (3,4,7,8)

THEOREM 4.6: *Let D be a PSI(α), m a nonnegative integer. Then*

- (9) $\det \bar{G}_m \geq 0$, with equality iff $\text{Pr}(D) \leq m$,
 (10) $\det \bar{H}_m \geq 0$, with equality iff $R(D) \leq m$.

As an example we discuss inequality (9) for $m = 1$. Then we get

COROLLARY 4.7: *Let D be a PSI(α). Then*

$$t_2 \geq (\sigma - \alpha)^2 (v-1)^{-1} + \alpha^2,$$

with equality iff D is an (r, λ) -design.

PROOF: $\bar{G}_1 = \begin{pmatrix} v-1 & \sigma-\alpha \\ \sigma-\alpha & t_2 - \alpha^2 \end{pmatrix}$ and by (1.1.) and the example in 4.3 we obtain

$\text{Pr}(D) \leq 1$ iff $\text{Pr}(D)=1$ iff D is an (r, λ) -design.

Using the number c_2 of proper two-gons (p, B, q, C, p) $p \neq q, B \neq C$ we get

COROLLARY 4.8: *Let D be a PSI(α, r). Then*

$$c_2 \geq (v-1)^{-1} v (\alpha - r) (\alpha - r - v + 1),$$

with equality iff D is an (r, λ) -design.

From Corollary 4.8 we may derive again the FISHER-inequality for 2-designs, the inequality 3.2 for partial geometric designs and the inequality for partial 1-designs $D : v-1 \geq r(k-1)$, with equality iff D is a $2-(v, k, 1)$ -design.

The following theorem sharpens corollary 4.7.

THEOREM 4.9: *Let D be a PSI(α). Then*

$$t_2 \geq (v-1)^{-1} (v\alpha^2 - 2\alpha\sigma + v \sum_{p \in P} \{p\}^2) \geq (\sigma - \alpha)^2 (v-1)^{-1} + \alpha^2,$$

the first equality holds iff D is an (r, λ) -design,

the second equality holds iff D is a $PSI(\alpha, r)$.

PROOF: From $t_2 = \sum_{p \in P} (w(p) + [p]^2)$ and Thm. 2.3. we obtain the first inequality.

The second inequality is equivalent to the variance-inequality (compare Thm. 2.1.)

$$\sum_{p \in P} [p]^2 \geq \sigma^2 v^{-1} \text{ with equality iff } [p] = \sigma v^{-1} \text{ for all points } p.$$

5. BOUNDS FOR THE CONNECTION NUMBERS OF REGULAR POINT STABLE DESIGNS

The following theorem generalizes and improves the inequalities of AGRAWAL [1,2] for the intersection numbers $[B,C]$ of different blocks B,C of a 1-design. The AGRAWAL-inequalities contain the SHAH-inequalities for PBIBD's [11], and the CONNOR-MAJUMDAR-inequalities for 2-designs [5,7].

THEOREM 5.1: Let D be a connected $PSI(\alpha, r)$,

$$\text{spec}N = \{\alpha, \rho_1, \dots, \rho_m\}, \alpha > \rho_1 > \dots > \rho_m.$$

Then for any two different points p, q we have

$$\begin{aligned} \max \{2r - b, r - \rho_1, 2(\alpha - \rho_m)v^{-1} + \rho_m - r\} &\leq [p, q] \leq \\ \min \{r - \rho_m, 2(\alpha - \rho_1)v^{-1} + \rho_1 - r\}. \end{aligned}$$

The proof will be given elsewhere.

COROLLARY 5.2: (AGRAWAL [1,2]): Let D be a connected $1-(v, k, r)$ -design,

$$\text{spec}N = \{rk, \rho_1, \dots, \rho_m\}, rk > \rho_1 > \dots > \rho_m.$$

Then for any two different blocks B, C we have

$$\begin{aligned} \max \{2k - v, k - \rho_1, 2rkb^{-1} - k\} &\leq [B, C] \leq \\ \min \{k, 2(rk - \rho_1)b^{-1} + \rho_1 - k\}. \end{aligned}$$

REMARK: It is easily seen that the application of Thm. 5.1. to the dual of the 1-design D in Cor. 5.2. yields exactly the AGRAWAL-bounds if $\det(A^T A) = 0$, where A is the incidence matrix of D , while otherwise both bounds

$2rkb^{-1} - k \leq [B, C] \leq k$ are improved by Thm. 5.1.

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