Math. Z. 183, 565–574 (1983)

Mathematische
Zeitschrift

On Norm Three Vectors in Integral Euclidean Lattices. I

Arnold Neumaier

Institut für Angewandte Mathematik, Universität Freiburg,
Hermann-Herder-Straße 10, D-7800 Freiburg, Federal Republic of Germany

1. Introduction

In this paper we look at the structure of the set of norm 3 vectors of an integral Euclidean lattice L. It can be thought of a beginning of the study of integral lattices generated by vectors of specified norm. Such lattices arise naturally in various contexts. By Dade [6], every finite subgroup of $GL(n,\mathbb{Z})$ preserves an integral lattice in $\mathbb{R}^n$ generated by minimum norm vectors. By Coxeter [5], many lattices of extreme forms are generated by their minimum norm vectors, and Barnes [1] shows that a lattice L is extreme if the sublattice generated by the minimum norm vectors of L spans $\mathbb{R}\otimes L$ and is extreme.

The integral lattices generated by vectors of norm 2 are known for a long time; they are the root lattices A_n $(n\geqq 1)$, D_n $(n\geqq 4)$, E_n $(n=6,7,8)$ – see Witt [14], Coxeter [4, 5]. In Cameron et al. [3], this fact was derived in a combinatorial way, and used to classify sets of lines at angles 90° and 60°, and of graphs with smallest eigenvalue $\geqq -2$. In fact, if A is the adjacency matrix of a graph Γ, i.e., $A=(m_{xy})_{x,y\in\Gamma}$ with $m_{xy}=1$ if xy is an edge and $m_{xy}=0$ otherwise, and if the smallest eigenvalue of A is $\geqq -m$, where m is an integer, then $mI+A$ is the Gram matrix of a set of vectors of norm m in Euclidean space, and these vectors generate an integral lattice. Often, and for $m=2$ always, this lattice has minimum norm m.

The integral lattices generated by vectors of norm 1 are trivially the lattices $\mathbb{Z}^n$; so the first unsettled case is that of norm 3. To study the set of norm 3 vectors we use methods developed by Shult and Yanushka [12], who generalized the combinatorial approach of Cameron et al. to study sets of lines at angles 90° and $\arccos\frac{1}{3}$. The connection is given by the fact that the vectors of norm 3 (=length $\sqrt{3}$) along these lines generate an integral lattice. In fact much of what we do here is an extension or variation of results in [12]. We also use this opportunity to point out some inaccuracies in [12]:

(i) Proposition 3.10 of [12] is correct only under the extra assumption that every pair of vectors $x,y\in\Sigma$ with $(x,y)=-1$ is in some tetrahedron of Σ (see the remark after Proposition 9 below).

ii) The proof of Proposition 3.14 of [12] contains a gap. It is not at all clear why the Gram matrix constructed is positive semidefinite, hence it remains dubious whether the line system constructed is not in indefinite space.

iii) Similarly, the definiteness of the Gram matrix for the line system with 40 lines ([12], p. 68ff.) is not shown.

In a sequel to this paper we shall correct the gap ii) at least for regular near hexagons, and give a different construction for a $(0, \frac{1}{3})$-line system in $\mathbb{R}^9$ with 40 lines, thus making the construction of [12] mentioned in iii) superfluous.

2. Lattices, Norm 3 Vectors, and Line Systems

A *lattice* L of dimension n is a free $\mathbb{Z}$-module with n (free) generators together with a real symmetric bilinear form which we write as (x, y). We call the number (x, x) the *norm* of $x \in L$. A lattice is *integral* if $(x, y) \in \mathbb{Z}$ for all $x, y \in L$, and *Euclidean if* $(x, x) \geqq 0$ for all $x \in L$. L is uniquely determined by a basis $p_1, \ldots, p_n$ together with its (symmetric) Gram matrix $G = ((p_i, p_j))_{i,j=1,\ldots,n}$; then we have $(\Sigma \alpha_i p_i, \Sigma \beta_i p_i) = \alpha^T G \beta$ where $\alpha = (\alpha_i)$, $\beta = (\beta_i)$. L is integral if and only if G is integral, and Euclidean if and only if G is positive definite.

More generally, if G is a symmetric, positive semidefinite $v \times v$-matrix then the columns $g_1, \ldots, g_v$ of G generate a lattice L of dimension $\leqq v$ such that $G = ((g_i, g_j))$ is the Gram matrix of the spanning set $g_1, \ldots, g_v$; the inner product of L is the restriction of the standard inner product of $\mathbb{R}^v$ to L.

From now on, all lattices considered will be Euclidean.

As in [12], we call a set S of lines in a Euclidean space an $(\alpha_1, \ldots, \alpha_s)$-*system* if any two unit vectors along distinct lines have inner product $\in \{\pm\alpha_1, \ldots, \pm\alpha_s\}$. We are interested in $(\frac{1}{3})$-systems, $(0, \frac{1}{3})$-systems, and $(0, \frac{1}{3}, \frac{2}{3})$-systems S. In this case it is more convenient to take vectors of norm 3 along the lines of S; the set Σ of all these vectors has integral inner products and satisfies $\Sigma = -\Sigma$. In particular, Σ generates an integral lattice.

Conversely, the set Γ of norm 3 vectors of an integral lattice L gives rise to $(0, \frac{1}{3}, \frac{2}{3})$-systems by taking the lines through vectors of Γ (or any subset Σ of Γ); indeed, for $x, y \in \Gamma$ we have $(x, y)^2 \leqq (x, x)(y, y) = 9$, hence $(x, y) \in \{0, \pm 1, \pm 2, \pm 3\}$, and $(x, y) = \pm 3$ implies $x = \pm y$. Loosely, we call Σ an $(\alpha_1, \ldots, \alpha_s)$-system if the corresponding set S has this property.

The case of lattices with minimum norm 3 is special:

Proposition 1. *Let L be an integral lattice with minimum norm* 3. *Then* $\Gamma = \{x \in L \mid (x, x) = 3\}$ *is a* $(0, \frac{1}{3})$-*system.*

Proof. If $x, y \in \Gamma$ and $(x, y) = \pm 2$ then $x \mp y$ is in L and has norm 2, contradiction. □

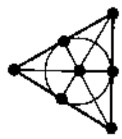
We now give examples of lattices with interesting sets of norm 3 vectors, and of $(\frac{1}{3})$-systems and $(0, \frac{1}{3})$-systems. In the examples, the vectors e_i are orthonormal, i.e. $(e_i, e_i) = 1$ and $(e_i, e_j) = 0$ for $i \neq j$.

Example 1. Let A be the adjacency matrix of a connected regular graph with v vertices and valency 3. Then 3 is a simple eigenvalue of A, and all other eigenvalues of A are <3; hence $3I-A$ is the Gram matrix of a $(0,\frac{1}{3})$-system in $\mathbb{R}^{v-1}$. Similarly, if the adjacency matrix A of a graph has smallest eigenvalue $\geqq -3$ then $3I+A$ is the Gram matrix of a $(0,\frac{1}{3})$-system.

Example 2. Let $L=\mathbb{Z}^n$, with basis $e_1,\ldots,e_n$. Then $\Gamma=\{\pm e_i \pm e_j \pm e_k | 1\leqq i<j<k\leqq n\}$. Γ is a $(0,\frac{1}{3},\frac{2}{3})$-system, but it contains interesting subsets forming $(0,\frac{1}{3})$-systems of $(\frac{1}{3})$-systems. In fact, if $\mathscr{B}$ is a *partial linear space* with lines of size 3, i.e. a set of 3-subsets (lines) of $\{1,\ldots,n\}$ such that two lines have at most one common point then

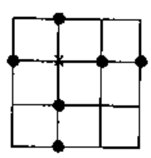
$$\Sigma_1(\mathscr{B}) := \{\pm e_i \pm e_j \pm e_k | \{i,j,k\}\in\mathscr{B}\}$$

is a $(0,\frac{1}{3})$-system. If $\mathscr{B}$ is one of the two spaces

B_6  B_7 

then $\Sigma_1(\mathscr{B})$ is a $(\frac{1}{3})$-system since any two lines intersect.

Example 3. Let $\mathscr{D}$ be a *design*, i.e. a set of subsets (blocks) of a set of points, with v points an b blocks of size 6, such that any two blocks intersect in 0 or 2 points. For each block $B\in\mathscr{D}$, let $\mathscr{C}_B$ be a self-complementary binary code (cf. McWilliams and Sloane [8]) of length 6 over $\{0,1\}$ such that no two codewords have Hamming distance one; e.g. $\mathscr{C}_B=\mathscr{C}_5$, the set of 20 words $(0^3 1^3)$ of weight 3, or $\mathscr{C}_B=\mathscr{C}_6$, the set of 32 words (0^6), $(0^4 1^2)$, $(0^2 1^4)$, (1^6) of even weight. Then the set $\Sigma_2(\mathscr{D})$ consisting of all vectors $\frac{1}{\sqrt{2}}\sum_{i\in B}(-1)^{\varepsilon_i} e_i$, where $B\in\mathscr{D}$, $(\varepsilon_i)_{i\in B}\in\mathscr{C}_B$, is a $(0,\frac{1}{3})$-system. If $\mathscr{D}$ is the $2-(16,6,2)$-biplane whose blocks are the 16 sets of shape



and $\mathscr{C}_B=\mathscr{C}_6$ for all blocks $B\in\mathscr{D}$ we obtain the line system S_{256} of [12]. If $\mathscr{D}$ consists of only one block we get the particular case $k=2$ of the next example.

Example 4. Let $\mathscr{C}$ be a self-complementary binary code of length $3k$ over $\{0,1\}$ such that any two codewords have Hamming distance $\in\{0,k,\frac{3}{2}k,2k,3k\}$. Then

$$\Sigma_3(\mathscr{C})=\left\{\frac{1}{\sqrt{k}}\Sigma(-1)^{\varepsilon_i} e_i | (\varepsilon_i)\in\mathscr{C}\right\}$$

is a $(0,\frac{1}{3})$-system, and a $(\frac{1}{3})$-system if $\frac{3}{2}k$ does not occur as distance. If $\mathscr{C}$(mod 2) is linear then the lattice $\frac{1}{\sqrt{k}}L(\mathscr{C})$, where $L(\mathscr{C})$ consists of all $\Sigma\alpha_i e_i\in\mathbb{Z}^{3k}$ with

(i) all α are congruent to the same value $\gamma \bmod 2$,
(ii) the vector $\frac{1}{2}(\alpha_i-\gamma) \bmod 2$ is in $\mathscr{C}$,
(iii) $\Sigma\, \alpha_i \equiv 0 \bmod n$,

has the set $\Sigma_3(\mathscr{C})$ as the set of norm 3 vectors. For example, if $\mathscr{C}=\mathscr{C}_{24}$ is the (linear) binary Golay Code of length 24 ([8], Chap. 20) we get a line system S_{2048} also considered in [12].

For further constructions see Neumaier [10]. We see that the structure of the set of norm 3 vectors is essentially more complex than that for norm 2.

3. The Geometry of Closed Subsets

From now on, L is an integral Euclidean lattice containing vectors of norm 3. We adopt the following convention. A *point* is a vector of norm 3 in L, i.e. an element of Γ. We call two points x, y adjacent if $(x,y)=-1$; this turns Γ into a graph. The 4-cliques of Γ are called *planes*.

Proposition 2. *The points on a plane add up to zero (i.e. they are the vertices of a regular tetrahedron). Every triangle is in a unique plane. Planes intersect in at most two points.*

Proof. Let $x\,y\,z$ be a triangle, and $w=-x-y-z$. Then $(w,x)=-3+1+1=-1$, and similarly $(w,y)=(w,z)=-1$. Hence $(w,w)=1+1+1=3$. So $w\in\Gamma$, and $x\,y\,z\,w$ is a plane. If $x\,y\,z\,w'$ is another plane then $(w',w)=1+1+1=3$, whence $w'=w$. □

Let us call a subset Σ of Γ *closed* if Γ contains with a triangle $x\,y\,z$ the plane through $x\,y\,z$. In particular, Γ itself is closed. Also, the intersection of closed subsets is closed.

Example (cf. [12], Prop. 3.7b). Let $L=\mathbb{Z}^n$. The planes of Γ are either

$$\{e_i+e_j+e_k,\, e_i-e_j-e_k,\, -e_i+e_j-e_k,\, -e_i-e_j+e_k\},$$

or

$$\{e_i+e_j+e_k,\, -e_i-e_p+e_q,\, -e_j-e_q+e_r,\, -e_k-e_r+e_p\},$$

or such a set with certain e_l replaced by $-e_l$. Hence a subset of Γ of the shape $\Sigma_1(\mathscr{B})$, as defined in Example 2, is closed if and only if $\mathscr{B}$ satisfies Pasch's axiom:

(P) The third points on the lines of a triangle of $\mathscr{B}$ form a line of $\mathscr{B}$.

There are many interesting partial linear spaces with line size 3 satisfying (P); a classification of these spaces would be very interesting. Known to me are the following classes of examples:

1. Partial linear spaces with line size 3 but without proper triangles.
2. Projective spaces over $GF(2)$, see e.g. Hirschfeld [7].
3. The spaces whose points are the edges of a complete graph K_m, and whose lines are the triangles of K_m; these are special cases of cotriangular graphs considered by Shult [11].
4. Some sporadic examples related to the sequence of groups $L_3(2)\subset U_3(3)\subset HJ\subset G_2(4)\subset Sz$; see Neumaier [9].

Clearly, the intersection of closed sets is closed. We mention another useful observation.

Proposition 3. *Let Δ and Σ be subsets of Γ, and suppose that Σ is closed. Then $\Delta^\Sigma = \{x \in \Sigma \mid (x,u) \equiv 1 \bmod 2$ for all $u \in \Delta\}$ is closed.*

Proof. If x, y, z is a triangle in Δ^Σ then $w = -x-y-z \in \Sigma$, and $(w,u) = -(x,u) -(y,u)-(z,u) \equiv -1-1-1 \equiv 1 \bmod 2$ for all $u \in \Delta$. Hence $w \in \Delta^\Sigma$. □

If we observe that a subset Δ of Γ is a $(\frac{1}{3})$-system iff $(u,w) \equiv 1 \bmod 2$ for all $u, v \in \Delta$, we find immediately the following

Corollary (cf. [12], Prop. 3.8). *A maximal $(\frac{1}{3})$-subsystem of a closed subset of Γ is closed.*

For a subset Σ of Γ, the Σ*-neighbourhood* of a point $u \in \Gamma$ is the set $\Sigma(u) = \{x \in \Sigma \mid (x,u) = -1\}$ consisting of all points of Σ adjacent with u. We have

Proposition 4 (cf. [12], Prop. 3.14). *Let Σ be a closed subset of Γ, and $u \in \Sigma$. Then*

(i) *$\Sigma(u)$ is a partial linear space with lines of size 3. If l is a line of $\Sigma(u)$, and $p \in \Sigma(u)$ is not on l then p is adjacent with at most one point of l.*

(ii) *The set $u^\Sigma = \{u, -u\} \cup \Sigma(u) \cup -\Sigma(u)$ is closed.*

(iii) *If Σ is a $(\frac{1}{3})$-system then $u^\Sigma = \Sigma$.*

Proof. (i) Lines are the triangles xyz such that $uxyz$ is a plane. Since two planes have at most two common points, two lines have at most one common point. If $l = xyz$ is a line, and $p \in \Sigma(u)$ is not on l but adjacent with two points of l, say x and y then $pxyu$ is a plane intersecting the plane $pxyz$ in three points, contradiction.

(ii) is the special case $\Delta = \{u\} \subseteq \Sigma$ of Proposition 3.

(iii) If Σ is a $(\frac{1}{3})$-system and $x \in \Sigma$ then either $(x,u) = \pm 3$ and $x = \pm u$ or $(x,u) = \pm 1$ and $\mp x \in \Sigma(u)$. Hence $x \in u^\Sigma$. Therefore $u^\Sigma = \Sigma$. □

Now we aim at the classification of closed $(\frac{1}{3})$-systems.

Proposition 5. *Let Δ be a $(\frac{1}{3})$-subsystem of Γ, and π be a plane of Δ. Then a point $p \in \Delta - \pi$ is adjacent with exactly two points of π.*

Proof. If $\pi = x_1 x_2 x_3 x_4$ then $x_1 + x_2 + x_3 + x_4 = 0$, hence also $(p, x_1) + (p, x_2) + (p, x_3) + (p, x_4) = 0$. Now $(p, x_i) = \pm 1$ for all i implies that we have $(p, x_i) = -1$ exactly twice. □

Proposition 6. *Let Δ be a closed $(\frac{1}{3})$-subsystem of Γ, and $u \in \Delta$. Then the neighbourhood $\Delta(u) = \Delta \cap \Gamma(u)$ is a (possibly degenerate) generalized quadrangle with line size 3, i.e. if l is a line of $\Delta(u)$ and $p \in \Delta(u)$ is not on l then p is adjacent with exactly one point of l.*

Proof. Immediate from Proposition 5. □

Now, by a wellknown result (see e.g. Thas [13]) a generalized quadrangle with line size 3 is one of the following

(i) A set of points and no line,

(ii) A set of lines with a common point,

(iii) unique generalized quadrangles with 9, 15, and 27 points.

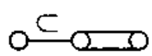
From this it is easy to deduce

Theorem 1 (cf. [12], Thm. 3.9). *A closed* $(\frac{1}{3})$*-subsystem* Δ *is isomorphic to one of the following sets (with orthonormal* e_i*):*

$$A_n = \{\pm(e_0 + \sqrt{2}\,e_i) \mid i = 1, \ldots, n\},$$
$$D_n = \{\pm e_0 \pm \sqrt{2}\,e_i \mid i = 1, \ldots, n\},$$
$$C_{10} = \Sigma_3(\mathscr{C}_5),$$
$$C_{16} = \Sigma_3(\mathscr{C}_6) \simeq \Sigma_1(\mathscr{B}_6),$$
$$C_{28} = \Sigma_1(\mathscr{B}_7).$$

Proof. A_n realizes (i), D_n realizes (ii), and C_{10}, C_{16}, C_{28} realize the three cases of (iii). Since by Proposition 4 (iii), $\Delta = \{u, -u\} \cup \{p, -p \mid p \in \Delta(u)\}$, the graph structure of $\Delta(u)$ determines all inner products of Δ, hence closed $(\frac{1}{3})$-systems with isomorphic $\Delta(u)$ are in fact isomorphic. Hence the above list is complete. □

Theorem 2. *Points, edges and planes of* C_{10}, C_{16}, *or* C_{28} *form a Buekenhout geometry with intersection property and diagram*



the geometry obtained by identifying each point x *with* $-x$ *has the same properties.*

Proof. For the axioms of a Buekenhout geometry see Buekenhout [2]. They are satisfied by Proposition 6 since the generalized quadrangles occuring in (iii) are nondegenerate. □

For closed $(0, \frac{1}{3})$-systems it is much more difficult to get a complete description, but under suitable hypothesis we can again get Buekenhout geometries. We begin with

Proposition 7. *Let* Σ *be a closed* $(0, \frac{1}{3})$*-system and* $u \in \Sigma$. *Then distance* $0, 1, 2$, *or* 3 *in* $\Sigma(u)$ *implies inner product* $3, -1, +1$, *or* 0, *respectively.*

Proof. Let l be a line of $\Sigma(u)$, and $p \in \Sigma(u)$ a point not on l. Then the inner product of p with points of l has the pattern $++-$ or $+00$ (since the sum over the plane $l \cup \{u\}$ is zero). Suppose now that x and y are at distance i. Clearly $i=0$ if and only if $(x, y) = 3$, and $i=1$ if and only if $(x, y) = -1$. If $i=2$ then there is a line xuv such that y is adjacent with u. Hence $(u, y) = -1$, the pattern of y must be $++-$, and therefore $(x, y) = +1$. If $i=3$ then there is a line xuv such that y has distance 2 from u and distance ≥ 2 from v. Hence $(u, y) = +1$, $(v, y) \neq -1$, and of course $(x, y) \neq -1$. Hence the pattern of y must be $+00$, and therefore $(x, y) = 0$. □

In an arbitrary partial linear space Ω, let l be a line, and p a point not on l. The collection of distances between p and points of l is called a *pattern*. If the minimal distance in a pattern is i then the other distances in that pattern are i

or $i+1$. We call Ω a *weak polygon* if the only patterns with distances ≤ 3 are $12\ldots2$ and $23\ldots3$. The proof of Proposition 7 then immediately implies

Proposition 8. *Let Σ be a closed $(0,\frac{1}{3})$-system and $u\in\Sigma$. Then $\Sigma(u)$ is a weak polygon with line size 3.*

A connected weak polygon of diameter at most three is called a *near hexagon.*

Proposition 9 (cf. [12], Prop. 3.10). *Let Σ be a closed $(0,\frac{1}{3})$-system and assume that every edge of Σ is in a plane of Σ. Then the following conditions are equivalent:*

(i) *Each $\Sigma(u)$ is a near hexagon.*
(ii) *Each $\Sigma(u)$ is connected and has diameter at most three.*
(iii) *In each $\Sigma(u)$, inner product $3, -1, +1, 0$ implies distance $0, 1, 2, 3$.*
(iv) *$(x,y)=1, x,y\in\Sigma \Rightarrow \Sigma(x)\cap\Sigma(y)$ contains no isolated point.*

Proof. By Proposition 8 and the definition of a near hexagon, (i) and (ii) are equivalent. (ii) and (iii) are equivalent by Proposition 7.

(iii)→(iv): Suppose that $(x,y)=1, x,y\in\Sigma$, and $z\in\Sigma(x)\cap\Sigma(y)$. Then $(-y,z)=1$ and $-y,z$ are both in $\Sigma(x)$. By (iii), there is a point $w\in\Sigma(x)$ adjacent with $-y$ and z. Let v be the third point on the line of $\Sigma(x)$ through w and z. Then $(-y,v)=1$, i.e. $v\in\Sigma(y)$. Hence $\Sigma(x)\wedge\Sigma(y)$ contains the edge zv, and z is not isolated.

(iv)→(iii): Suppose that $x,y\in\Sigma(u)$ so that $u\in\Sigma(x)\wedge\Sigma(y)$. If $(x,y)=1$ then u has a neighbour $v\in\Sigma(x)\cap\Sigma(y)$, and $x\,v\,y$ is a path of length 2 in $\Sigma(u)$. Hence x and y have distance 2. If $(x,y)=0$ then take a plane through the edge $x\,u$. Now $x\,v\,w$ is a line of $\Sigma(u)$, and for that line, y has the pattern $00+$. Therefore y has distance 2 from v or w, and distance 3 from x. □

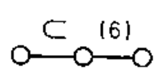
Remark. The condition that every edge is in a plane can not be deleted from Proposition 3.9. In fact, any $(0,\frac{1}{3})$-system without triangles (e.g. A_n of Theorem 1) is trivially closed and satisfies (iv), but $\Sigma(u)$ is a set of isolated points and not a near hexagon. This also shows that Proposition 3.10 of [12] is not valid as it stands (their (3.2) is equivalent to (iv)). In fact, in their proof they forget to check the condition on the diameter of the near hexagon.

A near hexagon without proper quadrangles is called a *generalized hexagon.* A set Σ of vectors is called *indecomposable* if there is no partition of Σ into nonempty subsets Σ_1 and Σ_2 such that $(x_1,x_2)=0$ for all $x_1\in\Sigma_1$, $x_2\in\Sigma_2$.

Theorem 3. *Let Σ be a indecomposable closed $(0,\frac{1}{3})$-system such that*

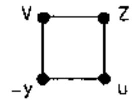
(0) *every edge is in a plane of Σ,*
(iv_1) *$(x,y)=1, x,y\in\Sigma \Rightarrow \Sigma(x)\cap\Sigma(y)$ is a disjoint union of edges.*

Then the points, edges and planes of Σ form a Buekenhout geometry with intersection property and diagram



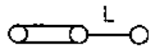
The same holds for the geometry obtained by identifying each point x with $-x$.

Proof. We have to show that each $\Sigma(x)$ is a generalized hexagon. In fact if



is a quadrangle in $\Sigma(x)$ then the third points u' and v' on the lines through z and u resp. v have inner product $+1$ with $-y$, and -1 with z; hence $\Sigma(x)\cap\Sigma(y)$ contains a path $u\,z\,v$ of length 2, contradiction. □

In Proposition 2.10 of [12], conditions are given for a near hexagon to be a Buekenhout geometry with diagram

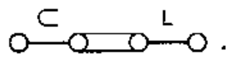


Together with Theorem 1 above, they imply

Theorem 4. *Let Σ be an indecomposable closed $(0,\frac{1}{3})$-system satisfying*

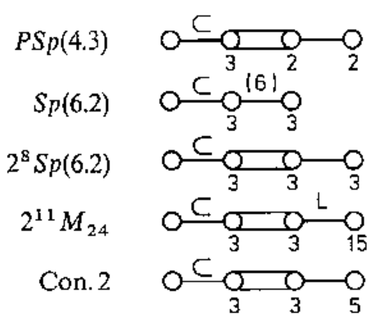
(0) *every edge is in a plane of Σ,*
(0_2) *every plane is in at least two maximal $(\frac{1}{3})$-subsets of Σ,*
(iv_2) $(x,y)=1,\ x,y\in\Sigma\Rightarrow\Sigma(x)\cap\Sigma(y)$ *has minimum valency* 2.

Then the points, edges, planes and maximal $(\frac{1}{3})$-subsets of Σ form a Buekenhout geometry with the intersection property and diagram



The same holds for the geometry obtained by identifying each point x with $-x$.

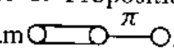
Examples. The line systems occuring in Theorem 1.3. of [12] satisfy the hypothesis of Theorem 3 or Theorem 4. This implies the existence of Buekenhout geometries with the following groups and diagrams:



Problems. 1. Classify the line systems satisfying the assumptions of Theorems 3 or 4.

2. Can one say something about $(0, \frac{1}{3})$-systems without triangles? One could consider closure under cubic graphs other than 4-cliques, e.g. the graph

$K_{3,3}$ 

where nonadjacent points shall be orthogonal. An analogue of Proposition 1 holds. Possibly we find Buekenhout geometries with diagram , on points, edges and $K_{3,3}$.

We close this section with some remarks on weak polygons in general.

Remarks. 1. Every near $2m$-gon (as defined in [12]) is a weak polygon.

2. A weak polygon may be quite irregular; e.g. it can be disconnected, and evem have isolated points.

3. Let Ω be a weak polygon. Then:

(i) Ω contains no triangles and no minimal pentagons.

(ii) Every subspace (line closed set) of Ω is a weak polygon.

(iii) Every connected subset of diameter 2 is contained in a subspace of diameter 2.

(iv) Every subspace of diameter 2 is a generalized quadrangle (possibly degenerate).

(v) If Q is a subspace of diameter 2, and $p \notin Q$ then one of the following holds:

(v_1) $Q \cup \{p\}$ has diameter 2,

(v_2) p is adjacent with at most one point of Q,

(v_3) Q is a complete bipartite graph, and p is adjacent to the points of some coclique of Q,

(v_4) Q contains no quadrangle.

The proofs given in [12] for near hexagons (Prop. 2.2, Lemma 2.3, Lemma 2.1, and a particular case of Prop. 2.6.) are valid in this more general case.

References

1. Barnes, E.S.: The construction of perfect and extreme forms. Acta Arithmetica **5**, 57–79 (1959)
2. Buckenhout, F.: Diagrams for geometries and groups. J. Combin. Theory Ser. A **27**, 121–151 (1979)
3. Cameron, P.J., Goethals, J.H., Seidel, J.J., Shult, E.E.: Line graphs, root systems, and elliptic geometry. J. Algebra **43**, 305–327 (1976)
4. Coxeter, H.S.M.: Discrete groups generated by reflections. Ann. of Math. **35**, 588–621 (1934)
5. Coxeter, H.S.M.: Extreme forms. Canad. J. Math. **3**, 391–441 (1951)
6. Dade, E.C.: The maximal finite groups of 4×4 integral matrices. Illinois J. Math. **9**, 99–122 (1965)
7. Hirschfeld, J.W.P.: Projective Geometries over Finite Fields. Oxford: Clarendon Press 1979
8. MacWilliams, F.J., Sloane, N.J.A.: The Theory of Error-correcting Codes. Amsterdam-New York: North Holland 1978

9. Neumaier, A.: Rectagraphs, diagrams, and Suzuki's sporadic simple group. Ann. Discrete Math. **15**, 305–318 (1982)
10. Neumaier, A.: Lattices of simplex type. SIAM J. Algebraic Discrete Methods (to appear)
11. Shult, E.E.: Groups, polar spaces and related structures. In: Combinatorics, Math. Centre Tracts **57**, 130–161 (1974)
12. Shult, E., Yanushka, A.: Near n-gons and line systems. Geom. dedicata **9**, 1–72 (1980)
13. Thas, J.A.: Combinatorics of partial geometries and generalized quadrangles. In: Higher Combinatorics (ed. M. Aigner). Proceedings of the NATO Advanced Study Institute (Berlin 1976), pp. 183–199. Dordrecht: Reidel 1977
14. Witt, E.: Eine Identität zwischen Modulformen zweiten Grades. Abh. Math. Sem. Univ. Hamburg **14**, 323–337 (1941)

Received December 13, 1982