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QUASI-RESIDUAL 2-DESIGNS, $1\frac{1}{2}$-DESIGNS, AND STRONGLY REGULAR MULTIGRAPHS*

ABSTRACT. It is proved that a quasi-residual 2-(v, k, λ)-design with $k > \frac{1}{2}\lambda^5 + O(\lambda^4)$ can be embedded into a symmetric 2-design. This improves a result by Bose, Shrikhande, and Singhi [1]. Our proof uses properties of strongly regular multigraphs and $1\frac{1}{2}$-designs. In particular, we give a simple sufficient condition for a strongly regular multigraph to be isomorphic to the block multigraph of a 2-design.

1. QUASI-RESIDUAL 2-DESIGNS

The paper is designed to simplify and improve results appearing in the paper 'Edge-regular multigraphs and partial geometric designs with an application to the embedding of quasi-residual designs' by Bose, et al. [1]. Since their paper is written in a very cumbersome notation, we keep this paper self-contained except for the following well-known results on designs (see e.g. Hall [3]).

A 2-(v, k, λ)-*design* consists of a set P of v points, a set $\mathscr{B}$ of blocks, and an incidence relation $\in$ between points and blocks such that every block contains (is incident with) k points, and every pair of distinct points is in (incident with) λ blocks. We refer to the structure as the 2-design $\mathscr{B}$ (on P). It is well-known that every point is in $r = \lambda(v-1)/(k-1)$ blocks, and the number of blocks in $\mathscr{B}$ is $b = \lambda v(v-1)/k(k-1)$. If we introduce the *order* $n = r - \lambda$ then the number of points becomes $v = k + \lambda^{-1}n(k-1)$. We also have Fisher's inequality $b \geqslant v$, with equality iff every pair of distinct blocks intersects in (is incident with) λ points. A 2-(v, k, λ)-design with $b = v$, i.e., $v = 1 + \lambda^{-1}k(k-1)$ is called a *symmetric* 2-*design*.

The *block multigraph* of a 2-design $\mathscr{B}$ is the multigraph whose vertices are the blocks, and two distinct vertices A, B are connected by m_{AB} edges, where m_{AB} is the number of points incident with A and B. In Section 3 we prove that the block multigraph of a 2-(v, k, λ)-design $\mathscr{B}$ of order n is a strongly regular multigraph $SR(m_0, n_0, \mu_0, \gamma_0, k_0)$ where

$$(1.1) \qquad m_0 = k, \qquad n_0 = n, \qquad \mu_0 = k^2\lambda, \qquad \gamma_0 = k(k-1)(\lambda-1),$$
$$k_0 = k(n + \lambda - 1).$$

In Section **4**, we prove a partial converse:

THEOREM 1.1. *Every strongly regular multigraph* $SR(m_0, n_0, \mu_0, \gamma_0, k_0)$,

* Part of this research was done while the author was at Westfield College, London.

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given by (1), *such that* n, k, λ *are positive integers*, $k \neq 1$, *and*

$$n > \max(k(k-1)\lambda^2 - (k-1)^2\lambda, 2(k-1)(k^2\lambda + k\lambda - 2\lambda + 1), \tfrac{1}{2}(k^2-1)(k^2\lambda - k + 2))$$

is isomorphic to the block multigraph of a 2-(v, k, λ)-*design with* $v = k + \lambda^{-1}n(k-1)$.

The proof of Theorem 1.1 involves more general designs, namely $1\frac{1}{2}$-designs (called partial geometric designs in [1]), and weak $1\frac{1}{2}$-designs (where we don't assume constant block size). Their importance can be seen, e.g., from the fact that 2-designs, dual 2-designs, transversal designs, semiregular partially balanced incomplete block designs, partial geometries, and polar spaces are examples of $1\frac{1}{2}$-designs. See Neumaier [6].

The block multigraphs of $1\frac{1}{2}$-designs, and, dually, the point multigraphs of weak $1\frac{1}{2}$-designs still have the property that they are strongly regular, and by investigating closely the properties of cliques and claws in a multigraph we obtain general characterization theorems which specialize to Theorem 1.1.

If H is a block of a symmetric 2-(v, k, λ)-design $\mathscr{B}$, we can define the *derived design* $\mathscr{B}_{\text{der}}$ and the *residual design* $\mathscr{B}_{\text{res}}$ (with respect to the block H). In both cases, the blocks are the elements of $\mathscr{B}$-$\{H\}$, and the points are the points in H for $\mathscr{B}_{\text{der}}$, and the points not in H for $\mathscr{B}_{\text{res}}$. Incidence is the same as before. In terms of the order n, the derived design is a 2-$(n+\lambda, \lambda, \lambda-1)$-design, and the residual design is a 2-(w, n, λ)-design with $w = \lambda^{-1}n(n+\lambda-1)$. Any 2-$(w, n, \lambda)$-design with $w = \lambda^{-1}n(n+\lambda-1)$ is called a *quasi-residual* 2-*design*. We want to find conditions which guarantee that a quasi-residual 2-design is the residual design of a symmetric 2-design; we call such designs *embeddable*.

For any two distinct blocks $A, B \neq H$, denote by m_{AB} the number of points in H incident with A and B, and by μ_{AB} the number of points not in H incident with A and B. Then $m_{AB} + \mu_{AB} = \lambda$, in particular $\mu_{AB} \leq \lambda$. Moreover, the multigraph on $\mathscr{B} - \{H\}$, with m_{AB} edges between A and B, is the block multigraph of $\mathscr{B}_{\text{der}}$. Hence, the residual design satisfies (i)–(iii) of the following theorem.

THEOREM 1.2. [1] *A quasi-residual* 2-(w, n, λ)-*design* $\mathscr{B}$ *is embeddable iff the following three conditions are satisfied:*

(i) *Any two distinct blocks* A *and* B *intersect in* $\mu_{AB} \leq \lambda$ *points,*

(ii) *The multigraph* $\mathscr{G}$ *on the blocks, with* $m_{AB} = \lambda - \mu_{AB}$ *edges between* A *and* B, *is a strongly regular multigraph* $SR(m, n, \mu, \gamma, k)$, *where*

$$m = \lambda, \qquad n, \quad \mu = \lambda^2(\lambda-1), \qquad \gamma = \lambda(\lambda-1)(\lambda-2), \qquad k = \lambda(n+\lambda-2), \tag{1.2}$$

(iii) $\mathscr{G}$ *is isomorphic to the block multigraph of a* 2-$(n+\lambda, \lambda, \lambda-1)$-*design* $\mathscr{B}'$.

To show that (i)–(iii) already imply embeddability we label the blocks of $\mathscr{B}'$ such that $\mathscr{B}' = \{\bar{B} \mid B \in \mathscr{B}\}$, and $\bar{A}$ and $\bar{B}$ intersect in m_{AB} points if $A \neq B$. Then define a new design whose points are those of $\mathscr{B}$ and $\mathscr{B}'$ together, and whose blocks are the sets $B \cup \bar{B}$, $B \in \mathscr{B}$, and one other block H, namely the point set of $\mathscr{B}'$. Using $m_{AB} + \mu_{AB} = \lambda$, it is easily verified that we get a symmetric 2-design which has $\mathscr{B}$ as residual design, and $\mathscr{B}'$ as derived design.

In Section 3 we show that (i) and (ii) of Theorem 1.2 are satisfied if

$$n > 2\lambda^3 - 4\lambda^2 + 4\lambda - 2. \tag{1.3}$$

If we combine this with Theorems 1.1 and 1.2 we find

THEOREM 1.3. *A quasi-residual* 2-(w, n, λ)-*design is embeddable if either* $\lambda = 3$, *and* $n > 76$, *or* $\lambda \neq 3$, *and* $n > \frac{1}{2}(\lambda^2 - 1)(\lambda^3 - \lambda^2 - \lambda + 2)$.

Theorem 1.2, and the remark before Theorem 1.3, appear already in [1]. There is also a theorem like our Theorem 1.1, with an inferior bound for n, and with a very complicated proof. They obtain Theorem 1.3 only under the stronger assumption $n > \lambda^5 + O(\lambda^4)$; so our method cuts down the bound to approximately half. For $\lambda < 3$, the restriction on n is superfluous, see Hall and Connor [4].

2. Strongly Regular Multigraphs

In this section, we define strongly regular multigraphs, look at their simplest properties, give necessary conditions on the parameters, and prove some results on cliques which will be useful in Section 4.

Let P be a finite set of *vertices*, or *points*. A *multigraph* on P, with *adjacency matrix* $M = (m_{ab})_{a,b \in P}$ is a collection $\mathscr{G}$ of *edges*, i.e., unordered pairs $\overline{ab}$ of distinct points such that $\overline{ab}$ appears m_{ab} times in $\mathscr{G}$. m_{ab} is called the *multiplicity* of $\overline{ab}$; $m_{ab} = m_{ba}$ is a nonnegative integer, and $m_{aa} = 0$ for all $a \in P$. We say that a, b are *adjacent* if $m_{ab} \geq 1$, and *nonadjacent* if $a \neq b$ and $m_{ab} = 0$. If $\mathscr{G}$ is a set, i.e., $m_{ab} \in \{0, 1\}$ for all $a, b \in P$, then $\mathscr{G}$ is a *graph*.

A *strongly regular multigraph* (short: *SR* multigraph) is a nonempty multigraph $\mathscr{G}$ on a v-set P whose adjacency matrix M satisfies the equations

$$MJ = kJ, \qquad M^2 = (n-2m)M + m(n-m)I + \mu J, \quad n > 0, \tag{2.1}$$

for some real numbers k, m, n, μ. (Here I is the identity matrix, and J is the all-one matrix of any size.)

In geometrical terms, this means that for all $a \in P$, the number of edges through a is constant,

$$\sum_x m_{ax} = k, \tag{2.2}$$

and, for all $a, b \in P$, the number of paths of length 2 between a and b depends linearly on m_{ab} and δ_{ab},

$$\sum_x m_{ax} m_{bx} = (n-2m)m_{ab} + m(n-m)\delta_{ab} + \mu. \tag{2.3}$$

(Here $\delta_{ab} = 1$ if $a = b$, $= 0$ otherwise, is the Kronecker symbol.) The number

$$\gamma = \sum_x m_{ax}(m_{ax} - 1) \tag{2.4}$$

is nonnegative, and zero iff $\mathscr{G}$ is a graph whence γ measures the deviation of $\mathscr{G}$ from a graph. γ is always even. By (2.2)–(2.4), we have

$$k = \sum_x m_{ax} = \sum_x m_{ax}^2 - \sum_x m_{ax}(m_{ax} - 1) = m(n-m) + \mu - \gamma. \tag{2.5}$$

Also, multiplying (2.1) by J, and observing $J^2 = vJ$, we find

$$v\mu = (k+m)(k+m-n). \tag{2.6}$$

In the situation (2.1)–(2.6), we say that $\mathscr{G}$ is a $SR(m, n, \mu, \gamma, k)$. We note that if m is an integer then n, μ, γ, k are nonnegative integers.

Remark. Our definition of a SR multigraph is a little more restrictive than the definition given in [1]. Our $SR(m, n, \mu, \gamma, k)$ is in their notation a SR multigraph $G_{k'}(r', d', \alpha_0', \ldots, \alpha_{r'}')$ with

$$k' = \frac{k}{m} + 1, \qquad r' = m, \qquad d' = \frac{\gamma}{2},$$

$$\alpha_i' = \mu - \frac{i}{m}(m(m-1) + \mu - \gamma).$$

The following Lemma is important for the construction (see Theorem 3.5) of the SR multigraph required in Theorem 1.2 (ii).

LEMMA 2.1. *Let M be an integral symmetric matrix with zero diagonal satisfying* (2.1).

Define $\gamma = m(n-m) + \mu - k$. *If*

$$n \geqslant \max(2m - 4, 2m - 1 + \mu + \gamma) \tag{2.7}$$

then M is the adjacency matrix of a SR multigraph SR (m, n, μ, γ, k).

Proof. Obviously (2.1) implies (2.3) and (2.4). Hence, we have for $a \neq b$ the following inequality:

$$\begin{aligned} 2m_{ab}(m_{ab} - 1) &\leqslant \sum_x (m_{ax} + m_{bx})(m_{ax} + m_{bx} - 1) \\ &= \sum_x m_{ax}(m_{ax} - 1) + 2\sum_x m_{ax}m_{bx} + \sum_x m_{bx}(m_{bx} - 1) \\ &= \gamma + 2((n-2m)m_{ab} + \mu) + \gamma. \end{aligned}$$

Therefore,

$$0 \leqslant (m_{ab} + 1)(m_{ab} + 2) \leqslant (n - 2m + 4)m_{ab} + \gamma + \mu + 2.$$

If (2.7) holds then this contradicts $m_{ab} \leqslant -1$; therefore m_{ab} is a nonnegative integer. Now the construction of the multigraph is obvious.

Equation (2.5) shows that the parameters of a SR multigraph are not independent. We list some more necessary conditions.

LEMMA 2.2. *Let $\mathscr{G}$ be a $SR(m, n, \mu, \gamma, k)$.*

(i) *$m - n \leqslant m_{ab} \leqslant m$, for all $a \neq b$.*
(ii) *$m \geqslant 1$, with equality iff $\mathscr{G}$ is the disjoint union of complete graphs.*
(iii) *If there are nonadjacent points then $n \geqslant m$.*
(iv) *$\mu \geqslant (k+m)(m-n)$, with equality iff $m_{ab} = m - n$ for all $a \neq b$.*
(v) *$\mu\gamma \leqslant (n - 2m + \mu)(m(n-m) + \mu)$, with equality if $\mathscr{G}$ contains no triangles.*
(vi) *$\mu \geqslant 2m - n$.*
(vii) *If $n \leqslant 2m + 4$ then $\gamma < 2m(n-m) + n - 2m - 1 + \mu$.*

Proof. (i) By (2.3), we have for $a \neq b$ the inequality

$$\begin{aligned} 2m_{ab}^2 \leqslant \sum_x (m_{ax} - m_{bx})^2 &= \sum_x m_{ax}^2 - 2\sum_x m_{ax}m_{bx} + \sum_x m_{bx}^2 \\ &= 2m(n-m) - 2(n-2m)m_{ab}, \end{aligned}$$

whence $(m - m_{ab})(m - n - m_{ab}) \geqslant 0$.

Since $n > 0$, this implies (i).

(ii) Since $\mathscr{G}$ is nonempty there is an edge $\overline{ab}$ whence $1 \leqslant m_{ab} \leqslant m$. If $m = 1$ then by (i), $\mathscr{G}$ is a graph. Then, if $m_{ab} = 1$, by the proof of (i), $m_{ax} = m_{bx}$ for all $x \neq a, b$; i.e., a and b are joined to exactly the same points. This implies that $\mathscr{G}$ is the disjoint union of complete graphs.

(iii) If there are nonadjacent points a, b then $m_{ab} = 0$, and (i) applies.

(iv) If $m_{ab} = m - n$ for all $a \neq b$ then, by (2.3), $\mu = (k+m)(m-n)$. Conversely, if we write $\mu = (k+m)(m-n) + s$ then by (2.2), (2.3) and (2.6),

$$0 \leqslant \sum_{x \neq a} \left(m_{ax} - \frac{k}{v-1}\right)^2 = \frac{s(nk - s)}{\mu(v-1)},$$

whence $s \geqslant 0$. Equality implies $m_{ax} = k/(v-1) = m - n$ for all $x \neq a$.

(v) By (2.2)–(2.5), the number of triangles containing a point a is

$$\begin{aligned} \sum_{x,y} m_{ax}m_{xy}m_{ya} &= \sum_x m_{ax}((n-2m)m_{ax} + \mu) \\ &= (n - 2m + \mu)(m(n-m) + \mu) - \gamma\mu; \end{aligned}$$

and this is obviously a nonnegative integer.

(vi) Follows from (v) by using $\gamma \geqslant 0$.

(vii) If M is the adjacency matrix of $\mathscr{G}$ then $M' = -M$ satisfies (2.1) and

(2.4) with

$$m' = n - m, \qquad n' = n, \qquad \mu' = -\mu, \qquad \gamma' = 2m(n-m) + 2\mu - \gamma, \qquad k' = -k,$$

but has some negative entries. Hence, the hypothesis of Lemma 2.1 cannot be satisfied. This leads to the stated condition.

A *clique* of a multigraph is a set of pairwise adjacent points. A clique which cannot be extended to a larger clique is called *maximal*.

In a $SR(m, n, \mu, \gamma, k)$, a maximal clique C with $|C| > (n/2) + \mu + 1 - m$ points is called a *grand clique*. This definition is motivated by the following lemma.

LEMMA 2.3. *An edge of multiplicity one is in at most one grand clique.*

Proof. Let $\overline{ab}$ be an edge of multiplicity one contained in two distinct grand cliques C and C'. Since C and C' are maximal there is $x \in C'$ such that $C \cup \{x\}$ is not a clique, and hence there is $y \in C$ with $m_{xy} = 0$. The points $z \in C \cap C'$ are adjacent to both x and y whence by (3), $|C \cap C'| \leqslant \sum_z m_{xz} m_{yz} = \mu$. Also, the points $z \in C \cup C' - \{a, b\}$ are adjacent to both a and b whence, by (2.3), $|C \cup C'| - 2 \leqslant \sum_z m_{az} m_{bz} = n - 2m + \mu$. Hence, $|C| + |C'| = |C \cap C'| + |C \cup C'| \leqslant n + 2(\mu + 1 - m)$. This contradicts the fact that both C and C' are grand cliques.

THEOREM 2.4. *If C is a clique of a $SR(m, n, \mu, \gamma, k)$ with $\mu > 0$ then*

$$|C|(k + m - \mu) \leqslant (n + 1 - m)(k + m). \tag{2.8}$$

Equality holds iff every edge contained in C has multiplicity 1, and, for every $x \notin C$, there are a constant number α of edges containing x and intersecting C; in this case, $\alpha = |C| + m - 1 - n$.

Proof. Let C be a clique with $|C| = c$ points. Define $\alpha_x = \sum_{a \in C} m_{ax}$. Then, for $x \notin C$, α_x is the number of edges containing x and intersecting C. We compute the expression

$$N(\alpha) = \sum_{x \notin C} (\alpha_x - \alpha)^2 + \sum_{x \in C} (\alpha_x - \alpha + m - n)(\alpha_x - \alpha + m). \tag{2.9}$$

Using (2.2)–(2.6), we find

$$\sum_x 1 = v, \tag{2.10a}$$

$$\sum_x \alpha_x = \sum_{a \in C} \sum_x m_{ax} = ck, \tag{2.10b}$$

$$\begin{aligned} \sum_x \alpha_x^2 &= \sum_{a,b \in C} \sum_x m_{ax} m_{bx} \\ &= \sum_{a,b \in C} ((n - 2m)m_{ab} + m(n-m)\delta_{ab} + \mu) \\ &= \sum_{a \in C} ((n - 2m)\alpha_a + m(n-m) + \mu c) \\ &= (n - 2m) \sum_{x \in C} \alpha_x + m(n-m)c + \mu c^2, \end{aligned} \tag{2.10c}$$

whence

$$\begin{aligned} N(\alpha) &= \sum_x (\alpha_x - \alpha)^2 + (2m - n) \sum_{x \in C} \alpha_x + (m(m-n) \\ &\quad - \alpha(2m - n)) \sum_{x \in C} 1 \\ &= \mu c^2 - c\alpha(2k + 2m - n) + \alpha^2 v \\ &= \mu^{-1}(c\mu - \alpha(k+m))(c\mu - \alpha(k + m - n)). \end{aligned}$$

In particular, for $\alpha = c\mu/(k+m)$, $N(\alpha) = 0$, and we conclude from (2.9) that $\alpha_x \leqslant \alpha + n - m$ for all $x \in C$ since otherwise $N(\alpha)$ would be strictly positive. But, for $x \in C$, $\alpha_x = \sum_{a \in C} m_{ax} \geqslant c - 1$ since C is a clique. Hence $c - 1 \leqslant \alpha + n - m$ which leads to (2.8).

If equality holds then $\alpha_x = c - 1 = \alpha + n - m$ for all $x \in C$. Hence, C contains only edges of multiplicity 1.

Moreover, $N(\alpha) = 0$ implies that $\alpha_x = \alpha = c + m - 1 - n$ for all $x \notin C$. Conversely, if C contains only edges of multiplicity 1, and $\alpha_x = \alpha'$ for all $x \notin C$ (for some α') then $\alpha_x = c - 1$ for all $x \in C$, and we obtain from (2.10a–c), after some calculation, that (2.8) holds with equality.

Note that we don't use this theorem for the proof of the results in Section 1.

3. $1\frac{1}{2}$-DESIGNS

A *design* consists of a set P of *points*, a set $\mathscr{B}$ of *blocks*, and an *incidence relation* $\in$ between points and blocks. Loosely, we say that $\mathscr{B}$ is a design on P, We use geometric language and say, e.g., that a point x is in (or on) a block B, or B contains x, if $x \in B$. A *flag* is an incident point-block pair. The *dual* of a design is obtained by interchanging the roles of points and blocks, and reversing the incidence relation. The *incidence matrix* of a design $\mathscr{B}$ on P is the matrix $A = (i_{aB})_{a \in P, B \in \mathscr{B}}$ with entries $i_{aB} = 1$ if $a \in B$, $= 0$ otherwise. The incidence matrix of the dual of $\mathscr{B}$ is then A^{T}.

A $1\frac{1}{2}$-*design* (called a *partial geometric design* in [1]) is a design on v points whose incidence matrix A satisfies the equations

$$AJ = rJ, \qquad JA = kJ, \qquad AA^{\mathrm{T}}A = nA + \alpha J, \tag{3.1}$$

for some integers r, k, n, α. In geometrical terms, this means that every block contains k points, every point is in r blocks, and, for any point x, and any block B, the number $\alpha(x, B)$ of flags $(a, A) \in P \times \mathscr{B}$ such that $a \neq x \in A \ni a \in B \neq A$

is

$$\alpha(x,B) = \sum_{A \neq B} \sum_{a \neq x} i_{xA} i_{aA} i_{aB} \tag{3.2}$$
$$= \begin{cases} \alpha & \text{if } x \notin B \\ \beta = n+1+\alpha-r-k & \text{if } x \in B. \end{cases}$$

Multiplying (3.1) on the left by J, we find

$$\alpha v = k(kr-n). \tag{3.3}$$

In the situation (3.1)–(3.3), we say that $\mathscr{B}$ is a $1\frac{1}{2}$-design with parameters (r, k, α, β, n).

The dual of a $1\frac{1}{2}$-design with parameters (r, k, α, β, n) is a $1\frac{1}{2}$-design with parameters (k, r, α, β, n).

LEMMA 3.1. (i) *Every 2-(v, k, λ)-design is a $1\frac{1}{2}$-design with parameters (r, k, α, β, n) given by*

$$r = \frac{\lambda(v-1)}{k-1}, \qquad k, \qquad \alpha = \lambda k, \qquad \beta = (\lambda-1)(k-1),$$
$$n = r - \lambda.$$

(ii) *Every $1\frac{1}{2}$-design with parameters (r, k, α, β, n) satisfying $\alpha = k(r-n)$ is a 2-(v, k, λ)-design with $\lambda = r-n$, $\lambda v = kr - n$.*

Proof. (i) The incidence matrix A of a 2-(v, k, λ)-design satisfies

$$AJ = rJ, \qquad JA = kJ, \qquad AA^{\top} = nI + \lambda J. \tag{3.4}$$

Multiplying the last equation by A we conclude (3.1); and β is computed by (3.2).

(ii) Let A be the incidence matrix of a $1\frac{1}{2}$-design $\mathscr{B}$ with $\alpha = k(r-n)$, and consider the symmetric matrix $X = AA^{\top} - nI - (r-n)J$. Straightforward calculations using (3.1) and (3.3), show that $X^2 = 0$, whence $X = 0$. Hence $AA^{\top} = nI + \lambda J$, with $\lambda = r - n$, so $\mathscr{B}$ is a 2-design. The expression for v is obtained by (3.3).

This result appears already in [1]. Also, it can be found in Neumaier [6], along with other results on $1\frac{1}{2}$-designs.

We weaken the definitions since we shall need designs with varying block sizes. A *weak 2-design* (or a *(r, λ)-design*) is a design whose adjacency matrix A satisfies the equations

$$AJ = rJ, \qquad AA^{\top} = nI + \lambda J, \qquad n = r - \lambda, \tag{3.5}$$

for some reals r, λ. A *weak $1\frac{1}{2}$-design* (called *gradkonstante SPG* by Wolff [8]) with parameters (v, n, r, λ) is a design on v points whose adjacency matrix A satisfies the equations

$$AJ = rJ, \qquad AA^{\top}A = nA + \lambda JA, \quad \lambda > 0, \tag{3.6}$$

for some reals r, n, λ.

Obviously, every 2-design is a weak 2-design, and a weak 2-design is a 2-design iff it has constant block size. Also, every $1\frac{1}{2}$-design with parameters (r, k, α, β, n) is a weak $1\frac{1}{2}$-design with

$$\lambda = \frac{\alpha}{k}, \qquad v = \frac{k(kr-n)}{\alpha},$$

and a weak $1\frac{1}{2}$-design is a $1\frac{1}{2}$-design if it has constant block size. Moreover, every weak 2-design is a weak $1\frac{1}{2}$-design.

The *point multigraph* $\mathscr{G}^{+}(\mathscr{B})$ of a design $\mathscr{B}$ has as vertices the points of $\mathscr{B}$, and two distinct points a, b are joined by as many edges as there are blocks through a and b. If every point is in exactly r blocks of $\mathscr{B}$, and A is the incidence matrix of $\mathscr{B}$ then $M = AA^{\top} - rI$ is the adjacency matrix of $\mathscr{G}^{+}(\mathscr{B})$. The *block multigraph* $\mathscr{G}_{+}(\mathscr{B})$ is the point multigraph of the dual. The importance of these concepts stems from

THEOREM 3.2. *The point multigraph of a weak $1\frac{1}{2}$-design with parameters (v, n, r, λ) is a SR multigraph $SR(m_0, n_0, \mu_0, \gamma_0, k_0)$, where*

$$\begin{aligned} &m_0 = r, \qquad n_0 = n, \qquad \mu_0 = \lambda(\lambda v + n), \\ &\gamma_0 = \lambda(\lambda-1)v - r(r-1) + (r+\lambda-1)n, \qquad k_0 = \lambda v - r + n. \end{aligned} \tag{3.7}$$

In particular, the point multigraph of a $1\frac{1}{2}$-design with parameters (r, k, α, β, n) is a SR multigraph $SR(m_0, n_0, \mu_0, \gamma_0, k_0)$ with

$$m_0 = r, \qquad n_0 = n, \qquad \mu_0 = \alpha r, \qquad \gamma_0 = \beta r, \qquad k_0 = r(k-1). \tag{3.8}$$

Proof. If we multiply (3.5) by J we find $AA^{\top}J = (\lambda v + n)J$ whence, with $M = AA^{\top} - rI$, $MJ = (\lambda v - r + n)J$. If we multiply (3.5) by $A^{\top}$ we find $(AA^{\top})^2 = nAA^{\top} + \lambda(AA^{\top}J)^{\top} = nAA^{\top} + \lambda(\lambda v + n)J$, whence $M^2 = (n-2r)M + r(n-r)I + \lambda(\lambda v + n)J$. Moreover, since $AA^{\top}$ is positive semidefinite, M has smallest eigenvalue $-r = -m_0$, whence $n = n_0 > 0$. Since M is the adjacency matrix of the point multigraph $\mathscr{G}^{+}(\mathscr{B})$, $\mathscr{G}^{+}(\mathscr{B})$ is strongly regular. The expression for γ_0 comes from Equation (2.5). The second part of the theorem is the specialization of the first part to $1\frac{1}{2}$-designs.

THEOREM 3.3 [1]. *The block multigraph of a 2-(v, k, λ)-design of order n is a SR multigraph $SR(m_0, n_0, \mu_0, \gamma_0, k_0)$, where*

$$m_0 = k, \; n_0 = n, \; \mu_0 = k^2\lambda, \; \gamma_0 = k(k-1)(\lambda-1), \; k_0 = k(n+\lambda-1). \tag{3.9}$$

Proof. By Lemma 3.1, the dual of a 2-(v, k, λ)-design of order n is a $1\frac{1}{2}$-design with parameters $(k, r = n + \lambda, \alpha = \lambda k, \beta = (\lambda-1)(k-1), n)$. Hence Theorem 3.2 applies.

If we apply Lemma 2.2(i) to this multigraph, we obtain the familiar result of Majumdar [5]:

COROLLARY 3.4 [5]. *Two distinct blocks A and B of a 2-(v, k, λ)-design intersect in at least $k - r + \lambda$ points.*

THEOREM 3.5 [1]. *Let $\mathscr{B}$ be a quasi-residual 2-(w, n, λ)-design with*

$$n \geqslant 2\lambda^3 - 4\lambda^2 + 4\lambda - 1. \tag{3.10}$$

Then two distinct blocks intersect in at most λ points, and property (ii) *of Theorem* 1.2 *holds.*

Proof. Since $\mathscr{B}$ is quasi-residual, $r = n + \lambda$, $b = r(r-1)/\lambda$. Hence, the incidence matrix A of $\mathscr{B}$ satisfies $AJ = nJ$, $JA = (n+\lambda)J$, $AA^\top = nI + \lambda J$. By straightforward calculations, the matrix $M = (n-\lambda)I + \lambda J - A^\top A$ satisfies Equation (2.1) with $m = \lambda$, n, $\mu = \lambda^2(\lambda - 1)$, $k = \lambda(n + \lambda - 2)$. Hence, with $\gamma = \lambda(\lambda-1)(\lambda-2)$, Lemma 2.1 applies. Therefore M is the adjacency matrix of a $SR(m, n, \mu, \gamma, k)$, i.e., (ii) of Theorem 1.2 holds. In particular, the off-diagonal entries $\lambda - \mu_{AB}$ of M are nonnegative, i.e., two distinct blocks A and B intersect in $\mu_{AB} \leqslant \lambda$ points.

The next two results are preliminary conditions for a SR multigraph to be the point multigraph of a weak $1\frac{1}{2}$-design, resp. a $1\frac{1}{2}$-design.

THEOREM 3.6. *A SR multigraph $SR(m, n, \mu, \gamma, k)$ is the point multigraph of a weak $1\frac{1}{2}$-design if and only if there is a collection Σ of cliques such that every point is in exactly m cliques of Σ, and every edge $\overline{ab}$ of multiplicity m_{ab} is in exactly m_{ab} cliques of Σ. In this case the blocks are the cliques of Σ, and the weak $1\frac{1}{2}$-design has parameters (v, m, r, λ) with*

$$v = \frac{(k+m)(k+m-n)}{\mu}, \qquad n, \qquad r = m, \qquad \lambda = \frac{\mu}{k+m}. \tag{3.11}$$

Proof. Let $\mathscr{G}$ be a $SR(m, n, \mu, \gamma, k)$. If $\mathscr{G}$ is the point multigraph of a weak $1\frac{1}{2}$-design $\mathscr{B}$ then the blocks of $\mathscr{B}$ are cliques in $\mathscr{G}$, and $\Sigma = \mathscr{B}$ satisfies the conditions of the theorem. Conversely, if Σ is a collection of cliques with the stated properties, then define a design $\mathscr{B}$ with Σ as set of blocks, and natural incidence. If A is the incidence matrix of $\mathscr{B}$, then the assumed properties can be stated in terms of A and the adjacency matrix M of $\mathscr{G}$ as $AJ = mJ$, $AA^\top = M + mI$.

With $\lambda = \mu/(k+m)$, the property that $\mathscr{G}$ is a $SR(m, n, \mu, \gamma, k)$ means $MJ = kJ$, $(M + mI)(M + (m-n)I - \lambda J) = 0$.

Hence $X = (AA^\top - nI - \lambda J)A$ satisfies $XX^\top = 0$, whence $X = 0$. Therefore, $AA^\top A = nA + \lambda JA$, and by Equation (2.6), $\mathscr{B}$ is a weak $1\frac{1}{2}$-design with parameters (3.11).

Remark. Wolff [8] proved a theorem similar to Theorem 3.6.

THEOREM 3.7. *Let $\mathscr{B}$ be a weak $1\frac{1}{2}$-design with parameters (v, n, r, λ), and $\lambda < 1$. If either*

(i) *two distinct points are in at most one block, or*
(ii) *$\alpha = \lambda(\lambda v + n)/r$ is an integer with $\lambda(n + 1 - r) < (1-\lambda)(\alpha + 1)$, then $\mathscr{B}$ is a $1\frac{1}{2}$-design, with parameters (r, k, α, β, n) given by*

$$r, k = \frac{\lambda v + n}{r}, \qquad \alpha = \frac{\lambda(\lambda v + n)}{r}, \qquad \beta = n + 1 + \alpha - r - k, \quad n. \tag{3.12}$$

Proof. Let B be a block. For any point x, the (x, B)-entry $s(x, B)$ of $AA^\top A$ is the number of flags $(a, A) \in P \times \mathscr{B}$ with $x \in A \ni a \in B$. If $x \in B$ then there are r possibilities with $a = x$, and $|B| - 1$ possibilities with $a \neq x$, $A = B$; and in case (i) no others. Hence $s(x, B) \geqslant r + |B| - 1$, where $|B|$ is the number of points in B. But since $AA^\top A = nA + \lambda JA$, and $x \in B$, we have $s(x, B) = n + \lambda|B|$. Therefore,

$$|B|(1-\lambda) \leqslant n + 1 - r. \tag{3.13}$$

In case (i), (3.13) is satisfied with equality, whence $\mathscr{B}$ has constant block size, and so is a $1\frac{1}{2}$-design.

If $x \notin B$ then $s(x, B) = \lambda|B| \leqslant \lambda(n + 1 - r)/(1-\lambda) < \alpha + 1$ by (3.13) and assumption (ii).
Since $s(x, B)$ and α are integers, $s(x, B) \leqslant \alpha$, or, with $k = \alpha/\lambda = (\lambda v + n)/r$:

$$|B| \leqslant k, \quad \text{for all blocks } B \in \mathscr{B}. \tag{3.14}$$

But $AA^\top J = (\lambda v + n)J = krJ$, whence, for any point x, $kr = \sum_B \sum_a i_{xB} i_{aB} = \sum_{B \ni x} |B| \leqslant rk$. This is only possible if $|B| = k$ for all $B \ni x$. Since x was arbitrary, $\mathscr{B}$ has constant block size, and so is a $1\frac{1}{2}$-design.

The computation of the parameters (3.12) is straightforward.

4. The Characterization Theorem

An *s-claw* is a pair (a, S) such that $m_{xy} = 0$ for $x, y \in S$, $m_{ax} \geqslant 1$ for $x \in S$, $\sum_{x \in S} m_{ax} = s$.

LEMMA 4.1. *Let $\mathscr{G}$ be a SR multigraph (m, n, μ, γ, k) with $\mu \geqslant 1$, and integral $m \geqslant 2$.*

(a) *If $2n > m(m-1)(\mu+1) + m\gamma + 2m - 2$ then, for every s-claw, $s \leqslant m$.*
(b) *If $2n > (m-3)(\mu - m) + 2\gamma + 2m - 2$ then every s-claw $(1 \leqslant s \leqslant m-2)$ can be extended to a $(s+1)$-claw.*
(c) *If there are no s-claws with $s > m$ then every $(m-1)$-claw is in $f_0 \geqslant n - 1 - (m-2)(\mu + 1 - m) - \gamma$ m-claws.*
(d) *If (a, S) is a maximal m-claw then there are $f_1 \geqslant m(n-2) - (m-2) \times \mu - 2\gamma$ m-claws (a, S') with $|S \cap S'| = m - 1$.*

Proof. Let (a, S) be a s-claw. Define $s_0 = |S|$. Then $s = \sum_{x \in S} m_{ax} \geqslant s_0$ since $m_{ax} \geqslant 1$ for $x \in S$.

Let us write $\alpha_x = \sum_{c\in S} m_{xc}$. Then $\alpha_x = 0$ if $x \in S$. We compute

$$\sum_{x\neq a} m_{ax} = k, \qquad \sum_{x\neq a} m_{ax}(m_{ax}-1) = \gamma,$$
$$\sum_{x\neq a} \alpha_x = \sum_{x\neq a}\sum_{c\in S} m_{xc} = \sum_{c\in S}(k - m_{ac}) = ks_0 - s,$$
$$\sum_{x\neq a} m_{ax}\alpha_x = \sum_{x\neq a}\sum_{c\in S} m_{ax}m_{cx} = \sum_{c\in S}(\mu + (n-2m)m_{ac}) = \mu s_0 + (n-2m)s,$$
$$\sum_{x\neq a} a_x^2 = \sum_{x\neq a}\sum_{c\in S}\sum_{d\in S} m_{xc}m_{xd} = \sum_{c\in S}\sum_{d\in S}(\mu + (n-2m)m_{cd} + m(n-m)\,\delta_{cd} - m_{ac}m_{ad}) = \mu s_0^2 + m(n-m)s_0 - s^2,$$

since $m_{cd} = 0$ for $c, d \in S$.

(a) We have

$$\begin{aligned} N &= \sum_{x\neq 0}(a_x - m_{ax})(a_x - m_{ax} - 1)\\ &= \sum_{x\neq a} a_x^2 - 2\sum_{x\neq a} m_{ax}a_x - \sum_{x\neq a} a_x + \sum_{x\neq a} m_{ax}(m_{ax}-1) + 2\sum_{x\neq a} m_{ax}\\ &= \mu s_0^2 + m(n-m)s_0 - s^2 - 2(\mu s_0 + (n-2m)s) - (ks_0 - s)\\ &\quad + \gamma + 2k, \end{aligned}$$

or, using $k = \mu + m(n-m) - \gamma$,

$$N = 2n(m-s) - s^2 + (4m+1)s - 2m^2 + \mu(s_0-1)(s_0-2) + \gamma(s_0-1). \tag{4.1}$$

Suppose now there are s-claws with $s \geqslant m+1$. Then let (a, S) be such that $s = m + t \geqslant m+1$ is minimal.

Since for $x \in S$, $(a, S - \{x\})$ is a $(s - m_{ax})$-claw, the minimality implies $m_{ax} \geqslant t \geqslant 1$ for all $x \in S$. Hence

$$N \geqslant \sum_{x\in S}(a_x - m_{ax})(a_x - m_{ax} - 1) = \sum_{x\in S} m_{ax}(m_{ax}+1) \geqslant (t+1)s. \tag{4.2}$$

On the other hand, $s = \sum_{x\in S} m_{ax} \geqslant ts_0$, whence $s_0 - 1 \leqslant m/t$. Therefore, (1) implies

$$\begin{aligned} N &\leqslant 2n(m-s) - s^2 + (4m+1)s - 2m^2 + \frac{\mu m(m-t)}{t^2} + \frac{\gamma m}{t},\\ N &\leqslant 2n(m-s) - s^2 + (4m+1)s - 2m^2 + \mu m(m-t) + \gamma m. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), and substituting $s = m + t$, we find

$$\begin{aligned} 2nt &\leqslant (\mu+1)m^2 + \gamma m - m(\mu-1)t - 2t^2\\ &\leqslant (\mu+1)m^2 + \gamma m - m(\mu-1) - 2\\ &= m(m-1)(\mu+1) + \gamma m + 2m - 2. \end{aligned}$$

Since $t \geqslant 1$, the hypothesis of (a) implies a contradiction. Hence, there are no s-claws with $s \geqslant m+1$.

(b) Assume $1 \leqslant s \leqslant m-2$, and denote by R the set of points x with $m_{ax} = 1$, $m_{cx} = 0$ for all $c \in S$. Then $\beta_x = m_{ax}(a_x - 1) + m_{ax}(m_{ax}-1)$ takes the value -1 if $x \in R$, and is nonnegative otherwise. Hence,

$$\begin{aligned} |R| \geqslant -\sum_{x\neq a}\beta_x &= -\sum_{x\neq a} m_{ax}\alpha_x + \sum_{x\neq a} m_{ax} - \sum_{x\neq a} m_{ax}(m_{ax}-1)\\ &= -\mu s_0 - (n-2m)s + k - \gamma\\ &\geqslant -(\mu + n - 2m)s + k - \gamma\\ &\geqslant -(\mu + n - 2m)(m-2) + k - \gamma\\ &= 2n - m - (\mu - m)(m-3) - 2\gamma > m - 2\\ &\geqslant s_0 = |S| \end{aligned}$$

(here we used Equation (2.5)). Hence, there is a point $x \in R\backslash S$, and $(a, S\cup\{x\})$ is a $(s+1)$-claw.

(c) Assume $s = m-1$, and denote by R' the set of points x with $m_{ax} \geqslant 1$, $m_{cx} = 0$ for all $x \in S$. Then, as before,

$$\begin{aligned} \sum_{x\in R} m_{ax} &\geqslant -\sum_{x\neq a} m_{ax}(a_x - 1) = -\mu s_0 - (n-2m)s + k\\ &\geqslant -(\mu + n - 2m)s + k = n - (m-2)(n-m) - \gamma. \end{aligned}$$

Obviously, $S \subseteq R'$, and if $x \in R'\backslash S$ then $(a, S\cup\{x\})$ is a s'-claw with $s' = m - 1 + m_{ax} \leqslant m$ by hypothesis of (c). Hence $m_{ax} = 1$. Therefore, the number of m-claws containing (a, S) is

$$\begin{aligned} |R'\backslash S| = \sum_{x\in R'} m_{ax} - \sum_{x\in S} m_{ax} &\geqslant n - (m-2)(\mu - m) - \gamma - s\\ &= n - 1 - (m-2)(\mu + 1 - m) - \gamma. \end{aligned}$$

(d) If (a, S) is a maximal m-claw then $a_x \geqslant 1$ for all $x \in S$. If we denote by Q the set of points x with $m_{ax} = 1, \alpha_x = 1$ then $\gamma_x = m_{ax}(a_x - 2) + m_{ax}(m_{ax}-1)$ takes the value -1 if $x \in Q$, is $\geqslant -2$ if $x \in S$, and is nonnegative otherwise. Hence,

$$|Q| \geqslant -2s_0 - \sum_{x\neq a}\gamma_x = -2s_0 - \sum_{x\neq a} m_{ax}\alpha_x + 2\sum_{x\neq a} m_{ax}$$

$$\begin{aligned} &-\sum_{x\neq a} m_{ax}(m_{ax}-1) = -2s_0 - \mu s - (n-2m)s + 2\gamma - \gamma \\ &\geqslant -(2+\mu+n-2m)s + 2k - \gamma \\ &= m(n-2) - (m-2)\mu - 2\gamma. \end{aligned}$$

This yields (d).

LEMMA 4.2. *Let $\mathscr{G}$ be a SR multigraph (m, n, μ, γ, k) with $\mu \geqslant 1$, and integral $m \geqslant 2$. If there are no s-claws with $s > m$, and if $n > \max(\frac{1}{2}(m-3)(\mu-m)+\gamma+m-1, 2(m-1)(\mu+1-m)+2\gamma)$ then each edge $\overline{ab}$ of multiplicity m_{ab} is in exactly m_{ab} grand cliques, and each point is in exactly m grand cliques.*

Proof. Let a be a point, and $\overline{ab}$ an edge of multiplicity $m_{ab} \geqslant 1$. By repeated application of Lemma 4.1(b), (c) the m_{ab}-claw $(a, \{b\})$ can be extended to a m-claw $(a, S \cup \{b\})$, with $|S| = m - m_{ab}$, $m_{ax} = 1$ for all $x \in S$. Then (a, S) is a $(m - m_{ab})$-claw, and can be extended to a m-claw $(a, S \cup S')$ with $|S'| = m_{ab}$, $m_{ax} = 1$ for all $x \in S'$, $S \cap S' = \varnothing$.

For $x \in S \cup S'$, define C_x as the set of all z with $m_{az} = 1$ for which $(S \cup S' - \{x\}) \cup \{z\}$ is a m-claw. By Lemma 4.1(c), $|C_x| \geqslant n - 1 - (\mu+1-m) \times (m-2) - \gamma > (n/2) + \mu - m$. Moreover, C_x is a clique since $z, z' \in C_x$, $m_{zz'} = 0$ would imply that $(S \cup S' - \{x\}) \cup \{z, z'\}$ is a $(m+1)$-claw. Since $m_{az} = 1$ for all $z \in C_x$, $\{a\} \cup C_x$ is a clique of size $> (n/2) + \mu + 1 - m$, and hence, by Theorem 1.1, contained in a unique grand clique C'_x. From the definition, the C_x are pairwise disjoint, hence the C'_x, $x \in S \cup S'$ are distinct.

We show now that $b \in C'_x$ iff $x \in S'$. In fact, if $x \in S$ then $m_{xb} = 0$ since $(a, S \cup \{b\})$ is a m-claw, hence $x \in C'_x$ implies $b \notin C'_x$. And if $x \in S'$, and $z \in C_x$ then $m_{bz} \geqslant 1$ since otherwise $(a, S \cup \{b, z\})$ would be a $(m+1)$-claw. Hence, $\{a, b\} \cup C_x$ is a clique containing $\{a\} \cup C_x$, and so contained in C'_x. In particular $b \in C'_x$.

Since $|S \cup S'| = (m - m_{ab}) + m_{ab} = m$, $|S'| = m_{ab}$, it remains only to show that the C'_x are the only grand cliques containing a. In fact, let C be any grand clique containing a. If $C \cap C_x \ni z$ then Lemma 2.3 implies (since $m_{az} = 1$) that $C = C'_x$. If $C \cap C_x = \varnothing$ for all $x \in S \cup S'$, then $C - \{a\}$ is contained in the set T of points adjacent to a but not in one of the C_x, $x \in S \cup S'$. In particular, since C is grand, $|T| \geqslant |C| - 1 > (n/2) + \mu - m > m(\mu+1-m) + \gamma - 1$.

But, by Lemma 4.1(c), the (disjoint) union of the sets $C_x - \{x\}$, $x \in S \cup S'$ has at least $f_1 \geqslant m(n-2) - \mu(m-2) - 2\gamma$ points. Since there are, at most, $k - m$ points adjacent to a, and not in $S \cup S'$, T has at most $(k-m) - (m(n-2) - \mu(m-2) - 2\gamma) = (\mu - m)(m-1) + \gamma \leqslant m(\mu+1-m) + \gamma - 1$ points, contradiction.

In [1], results like Lemmas 4.1 and 4.2 are obtained after complicated computations. Using these results, Wolff [8] proves a lemma similar to

LEMMA 4.3. *If $\mathscr{G}$ is a SR multigraph $SR(m, n, \mu, \gamma, k)$ with $\mu \geqslant 1$, integral $m \geqslant 2$, and $n > \max(2(m-1)(\mu+1-m)+2\gamma, \frac{1}{2}m(m-1)(\mu+1)+\frac{1}{2}m\gamma+m-1)$ then $\mathscr{G}$ is the point multigraph of a weak $1\frac{1}{2}$-design with parameters (v, n, r, λ) given by*

$$v = \frac{(k+m)(k+m-n)}{\mu}, n, \qquad r = m, \qquad \lambda = \frac{\mu}{k+m}. \tag{4.4}$$

Proof. Apply Lemmas 4.1 (a), 4.2 and Theorem 3.6 with the set of grand cliques as Σ, and observe that

$$\begin{aligned} &\tfrac{1}{2}(m-3)(\mu-m)+\gamma+m-1 \\ &\geqslant \frac{m(m-1)}{2}(\mu+1)+m\frac{\gamma}{2}+m-1. \end{aligned}$$

From Theorem 3.7, we obtain now an improvement of [1], Theorem 3.4:

THEOREM 4.4. *If $\mathscr{G}$ is a SR multigraph $SR(m, n, \mu, \gamma, k)$ with integral $m \geqslant 2$, integral $\mu \equiv 0 \bmod m$, $\mu > 0$, and*

$$n > \max\left(m-1+\frac{(\mu+m)\gamma}{m^2}, 2(m-1)(\mu+1-m)+2\gamma, \frac{m(m-1)}{2}(\mu+1)+m\frac{\gamma}{2}+m-1\right)$$

then $\mathscr{G}$ is the point multigraph of a unique $1\frac{1}{2}$-design, with parameters $(r', k', \alpha', \beta', n')$ given by

$$r' = m, \qquad k' = \frac{k}{m}+1, \qquad \alpha' = \frac{\mu}{m}, \qquad \beta' = \frac{\gamma}{m}, \qquad n' = n. \tag{4.5}$$

Proof. Apply Theorem 3.7 (ii) to the weak $1\frac{1}{2}$-design of Lemma 4.3. $\lambda < 1$ is satisfied by Equation (2.5) and the hypothesis, and condition (ii) is equivalent to $\mu \equiv 0 \bmod m$ and $n > m - 1 + ((\mu+m)\gamma/m^2)$. (The computations use Equation (4.4) above and Equation (2.5).) It is easy to see that, under the hypothesis, every block of a $1\frac{1}{2}$-design with point multigraph $\mathscr{G}$ is a grand clique; therefore, the $1\frac{1}{2}$-design is unique.

COROLLARY 4.5. *Theorem 1.1 is true.*

Proof. Apply Theorem 4.4 to the hypothesis of Theorem 1.1, and apply the dual of Lemma 3.1 (ii) to the resulting $1\frac{1}{2}$-design.

In particular, this completes the proof of Theorem 1.3. Note that the proof of Theorem 1.1 involves only Lemmas 2.3, 3.1 (ii), 4.1–4.3 and Theorems 3.6, 3.7 (ii), and 4.4, and the proof of Theorem 1.3 involves only Theorems 1.1, 1.2 and 3.5, and Lemma 2.1.

We conclude the paper with the specialization of Lemma 4.3 to graphs. A *strongly regular graph* is a strongly regular multigraph with $\gamma = 0$. A *partial geometry* is a $1\frac{1}{2}$-design with $\beta = 0$; see Bose [2].

In the situation of Lemma 4.3 with $\gamma = 0$, all edges have multiplicity one, and the blocks of the weak $1\frac{1}{2}$-design are grand cliques (by construction). Hence, by Lemma 2.3, two distinct points are on at most one block, and we may apply Theorem 3.7 (i). The condition $\lambda < 1$ is equivalent with $m < n + 1$. Hence we have

THEOREM 4.6. *If $\mathscr{G}$ is a strongly regular graph $SR(m, n, \mu, o, k)$ with $\mu \geqslant 1$, integral $m, 2 \leqslant m \leqslant n$, and*

$$n > \max\left(2(m-1)(\mu+1-m), \frac{m(m-1)}{2}(\mu+1)+m-1\right)$$

then $\mathscr{G}$ is the point graph of a unique partial geometry with parameters

$$r' = m, \qquad k' = \frac{k}{m} + 1, \qquad \alpha' = \frac{\mu}{m}.$$

This generalizes a classical result of Bose [2] for pseudo-geometric graphs, and can be used as a nonexistence theorem for not pseudo-geometric graphs, cf. Neumaier [7].

BIBLIOGRAPHY

1. Bose, R. C., Shrikhande, S. S., and Singhi, N. M.: 'Edge Regular Multigraphs and Partial Geometric Designs with an Application to the Embedding of Quasi-residual Designs'. *Coll. Int. Sul. Teo. Combin*, Tom I, 49–81, Acc Naz. Lincei, Roma, 1976.
2. Bose, R. C.: 'Strongly Regular Graphs, Partial Geometries, and Partially Balanced Designs'. *Pac. J. Math.* 13 (1963), 389–419.
3. Hall, M. jr.: *Combinatorial Theory*, Waltham, Mass., 1967.
4. Hall, M. jr. and Connor, W. S.: 'An Embedding Theorem for Balanced Incomplete Block Designs'. *Can. J. Math.* **6** (1953), 35–41.
5. Majumdar, K. N.: 'On Some Theorems in Combinatorics Relating to Incomplete Block Designs'. *Ann. Math. Statist.* **24** (1953), 379–389.
6. Neumaier, A.: '$t\frac{1}{2}$-designs'. *J. Comb. Theory* (Ser. A) **28** (1980), 226–248.
7. Neumaier, A.: 'Strongly Regular Graphs with Smallest Eigenvalue-*m*' *Arch. Math.* **33** (1979), 392–400.
8. Wolff, K. E.: 'Punktstabile und Semipartialgeometrische Inzidenzstrukturen'. *Mitt. Math. Sem.*, Giessen, 135 (1978).

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