KLEINE MITTEILUNGEN

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A Better Estimate for Fixed Points of Contractions

A well-known theorem in numerical analysis (see e.g. [1]) states that if $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous map with Lipschitz constant $\beta \in (0, 1)$, i.e.

$$\|\varphi(x) - \varphi(y)\| \leqq \beta \|x - y\| \quad \text{for} \quad x, y \in \mathbb{R}^n, \tag{1}$$

then the iteration

$$x_{i+1} := \varphi(x_i), \quad i = 0, 1, 2, \ldots; \quad x_0 \in \mathbb{R}^n \text{ arbitrary}, \tag{2}$$

converges, the limit $\hat{x}$ is a fixed point of φ, $\varphi(\hat{x}) = \hat{x}$, and the error estimates

$$\|\hat{x} - x_{n+i}\| \leqq \frac{\beta^{i+1}}{1-\beta} \|x_n - x_{n-1}\| \tag{3}$$

hold for $i \geqq 0$, $n \geqq 1$. Here $\|\cdot\|$ is an arbitrary norm of $\mathbb{R}^n$.

In this note we point out that we can do better if (1) holds with the *Euclidean norm*

$$\|x\|_2 = \sqrt{x^\mathsf{T} x}. \tag{4}$$

This is possible since the Euclidean norm has the following property:

Lemma: *For the Euclidean norm* (4), *the equivalence*

$$\|u + \beta^2 v\|_2 \leqq \beta \|u + v\|_2 \Leftrightarrow \|u\|_2 \leqq \beta \|v\|_2$$

holds for all $u, v \in \mathbb{R}^n$, *and all* $\beta \in (0, 1)$.

Proof: By (4), $\|u + \beta^2 v\|_2 \leqq \beta \|u + v\|_2$ holds iff

$$(u + \beta^2 v)^\mathsf{T} (u + \beta^2 v) \leqq \beta^2 (u + v)^\mathsf{T} (u + v)$$

iff

$$u^\mathsf{T} u + \beta^2 (u^\mathsf{T} v + v^\mathsf{T} u) + \beta^4 v^\mathsf{T} v \leqq \beta^2 u^\mathsf{T} u + \beta^2 (u^\mathsf{T} v + v^\mathsf{T} u) + \beta^2 v^\mathsf{T} v$$

iff

$$(1 - \beta^2)\, u^\mathsf{T} u \leqq \beta^2 (1 - \beta^2)\, v^\mathsf{T} v$$

iff $u^\mathsf{T} u \leqq \beta^2 v^\mathsf{T} v$ iff $\|u\|_2 \leqq \beta \|v\|_2$.

Using this result, we establish the following theorem:

Theorem: *For all* $n \geqq 1$, *we have the error estimates*

$$\|\hat{x} - z_n\|_2 \leqq \frac{\beta}{1 - \beta^2} \|x_n - x_{n-1}\|_2 \quad (\textit{Euclidean norm}) \tag{5}$$

for the sequences of vectors defined by (2) *and*

$$z_n := \frac{x_n - \beta^2 x_{n-1}}{1 - \beta^2} \tag{6}$$

Proof: Define $u = \hat{x} - z_n$, $v = (1 - \beta^2)^{-1} (x_n - x_{n-1})$. From (6) we find $u + v = \hat{x} - x_{n-1}$, $u + \beta^2 v = \hat{x} - x_n$, whence

$$\|u + \beta^2 v\|_2 = \|\varphi(\hat{x}) - \varphi(x_{n-1})\|_2 \leqq \beta \|\hat{x} - x_{n-1}\|_2 = \beta \|u + v\|_2.$$

By the lemma, $\|u\|_2 \leqq \beta \|v\|_2$ which implies (5).

Now we compare the error bounds. The bound in (3) is $\beta^i(1 + \beta)$ times the bound in (5). Hence the use of z_n saves i steps of the iteration (2) provided that $\beta^i(1 + \beta) \geqq 1$. To get a simple estimate of i we put $\varepsilon = 1 - \beta$. Then $\varepsilon > 0$ and $\beta = 1 - \varepsilon$, whence

$$\beta^i(1 + \beta) = (1 - \varepsilon)^i (1 + \beta) \geqq (1 - i\varepsilon)(1 + \beta) \geqq 1$$

provided that $i\varepsilon(1 + \beta) \leqq \beta$, or $i \leqq \beta/(1 - \beta^2)$. Hence we have:

Corollary: *The approximation* z_n *given by* (6) *has at least the accuracy guaranteed by* (3) *for the iterate* x_{n+i}, *where* $i := [\beta/(1 - \beta^2)]$.

In particular, z_n is always a better approximation to $\hat{x}$ than x_n. To see the dependence of i on β we give a small table:

β	$\leqq 0.6$	0.7	0.8	0.9	0.95	0.97	0.99
i	0	1	2	4	9	16	49

Thus, if β is close to one, we save a considerable number of iterations if we want to obtain a specified accuracy.

Remarks:

1. Of course, the above remains valid, with obvious changes in the proofs, for contraction mappings in any real or complex Hilbert space.

2. Since $z_1 = (1 - \beta^2)^{-1} (\varphi(x_0) - \beta^2 x_0)$, the preceding results might lead one to consider the iteration

$$y_{i+1} := \frac{\varphi(y_i) - \beta^2 y_i}{1 - \beta^2}, \quad i = 0, 1, 2, \ldots. \tag{7}$$

But this iteration may be divergent even if (1) holds! For example, the one-dimensional map φ with $\varphi(x) = -\beta x$, $\beta \in (0, 1)$ satisfies (1), and the iteration (7) becomes $y_{i+1} = -(1 - \beta)^{-1} \beta y_i$. Unless we start with the fixed point $y_0 = 0$, we obtain for $\beta \geqq \frac{1}{2}$ a divergent sequence.

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Generalised Root Iterations for the Simultaneous Determination of Multiple Complex Zeros

1. Introduction

An iterative scheme for inclusion of multiple complex zeros of a polynomial is described by Gargantini in [2]. The convergence of this algorithm is quadratic or cubic, depending on the number of wanted zeros. In [3] a parallel square-root iteration for multiple complex zeros is presented by the same author. The order of convergence of this method is four. Characteristics of these methods are: (i) both iterative procedures are formulated in complex circular arithmetic; (ii) an initial circular region is required for each of the zeros to be determined; (iii) the automatic determination of error bounds is possible for all given approximations at each iteration.

The aim of this paper is to present two generalised interval methods of the root iterations type for the simultaneous improving simple or multiple zeros of a (complex) polynomial P. The convergence order of these iterative procedures is $k + 2$

$(k \in N)$, where k is the order of the highest derivative of P used in the procedures. These interval processes are based on the root iterations for simultaneous finding simple complex zeros of the $(k+2)$-nd order (see [8]) and can be regarded as their extension. As special cases, for $k = 1$ and $k = 2$, from one of these generalised methods, the above cited interval methods of third and fourth order can be obtained. Generalised root iterations for multiple complex zeros have been described in [5]. The real case is given in [7].

In the sequel we shall use the following notation: P denotes the given *monic polynomial*, n its *degree* and ξ_i $(i = 1, \dots, q;$ $q \leq n)$ the *distinct zeros* of P of *multiplicity* μ_i $(\mu_1 + \dots + \mu_q = n)$. Further, we shall introduce the rational function $z \mapsto H_k(z)$ by

$$H_k(z) = \frac{(-1)^{k-1}}{(k-1)!} \frac{\mathrm{d}^k}{\mathrm{d}z^k} [\log P(z)], \qquad k \in N.$$

For a closed disc $Z = \{z: |z - c| \leq r\}$ in the complex plane with center c and radius r, we shall adopt the following notation: $Z = \{c; r\}$. The basic properties of circular arithmetic can be found in [1] or [8]. The k-th root of a disc, say $Z = \{c; r\}$, $c = |c| \exp(i\theta)$, $|c| > r$, which does not contain the origin, is defined as in [6]:

$$Z^{1/k} = \bigcup_{\lambda=0}^{k-1} \left\{ |c|^{1/k} \exp\left(i \frac{\theta + 2\lambda\pi}{k}\right); |c|^{1/k} - (|c| - r)^{1/k} \right\}. \qquad (1)$$

2. Interval Methods for Multiple Zeros

Consider a polynomial P of degree n

$$P(z) = \prod_{i=1}^{n} (z - \xi_i)$$

with simple or multiple complex zeros $\xi_1, \dots, \xi_q$, $q \leq n$, of multiplicity $\mu_1, \dots, \mu_q$, respectively. Assume we have found q nonoverlapping initial discs $Z_i^{(0)} = \{z_i^{(0)}; r_i^{(0)}\}$ such that $\xi_i \in Z_i^{(0)}$, $i = 1, \dots, q$. Let m be the iteration index. Introduce the following notations:

$$r^{(m)} = \max_{1 \leq j \leq q} r_j^{(m)};$$

$$\varrho^{(m)} = \min_{\substack{i,j \\ i \neq j}} \{|z_i^{(m)} - z_j^{(m)}| - r_j^{(m)}\};$$

$$\mu = \min_{1 \leq j \leq q} \mu_j;$$

$$\beta(k, n) = \begin{cases} 2n, & k = 1 \\ k(n - \mu), & k > 1 \end{cases};$$

$$\theta(k, n) = \begin{cases} 3n, & k = 1; \\ k(n - \mu), & k > 1 \end{cases};$$

$$c_i^{(m)} = \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{\mu_j}{(z_i^{(m)} - z_j^{(m)})^k};$$

$$\eta^{(m)} = \frac{\beta(k, n)\, r^{(m)}}{\varrho^{(m)k+1}}.$$

In the case of simple zeros, i.e. when $q = n$ and $\mu_1 = \mu_2 = \dots = \mu_n = \mu = 1$, under the condition

$$\varrho^{(0)} > \beta(k, n)\, r^{(0)} \qquad (2)$$

in [8] it was proved for each $i = 1, \dots, n$ and $m = 0, 1, \dots$:

$$|H_k(z_i^{(m)}) - c_i^{(m)}| > \eta^{(m)}, \qquad (3)$$

$$\left(\frac{1}{z_i^{(m)} - Z_j^{(m)}}\right)^k \subset \left\{ \frac{1}{(z_i^{(m)} - z_j^{(m)})^k}; \frac{k r^{(m)}}{\varrho^{(m)k+1}} \right\}. \qquad (4)$$

Starting from the initial inclusion discs $Z_i^{(0)}$ containing the zeros ξ_i, $i = 1, \dots, n$, the generalised interval method for the simultaneous finding of simple complex zeros of P is given in [8]:

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{[H_k(z_i^{(m)}) - \{c_i^{(m)}; \eta^{(m)}\}]_*^{1/k}}, \qquad (5)$$

$i = 1, \dots, n; \qquad k \in N; \qquad m = 0, 1, \dots$

According to (1) it follows that the k-th root of a disc is the union of k closed discs. A criterion for the choice of the appropriate k-th root set is considered in [8]. In the sequel the abbreviation CCR (criterion for choice of root) will be used for this criterion. Consequently, the symbol * in (5) denotes the disc satisfying CCR.

The following assertions for simple zeros are proved in [8]:

Theorem 1: *Let the sequences of intervals* $(Z_i^{(m)})$, $i = 1, \dots, n$, *are defined by* (5). *Then, under the condition* (2), *for each* $i = 1, \dots, n$ *and* $m = 0, 1, \dots$, *we have:*

1° $\xi_i \in Z_i^{(m)}$;

2° *the convergence order of the interval process* (5) *is* $k + 2$, $k \in N$.

Assume now that the zeros ξ_i are not necessarily simple, i.e. $q \leq n$. It is easy to show that

$$H_k(z) = \sum_{j=1}^{q} \frac{\mu_j}{(z - \xi_j)^k}$$

holds. Hence

$$(z - \xi_i)^k = \frac{\mu_i}{H_k(z) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{q} \mu_j \left(\frac{1}{z - \xi_j}\right)^k}$$

or

$$\xi_i = z - \frac{1}{\left\{ \frac{1}{\mu_i} \left[H_k(z) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{q} \mu_j \left(\frac{1}{z - \xi_j}\right)^k \right] \right\}^{1/k}}.$$

Let z be the center of the initial disc $Z_i^{(0)}$, i.e. $z = z_i^{(0)}$. Since $\xi_j \in Z_j^{(0)}$, $j = 1, \dots, q$, according to the inclusion monotonicity property we obtain

$$\xi_i \in z_i^{(0)} - \frac{1}{\left\{ \frac{1}{\mu_i} \left[H_k(z_i^{(0)}) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{q} \mu_j \left(\frac{1}{z_i^{(0)} - Z_j^{(0)}}\right)^k \right] \right\}_*^{1/k}} = Z_i^{(1)},$$

where the symbol * denotes the disc which satisfies CCR, i.e. the disc containing the complex number $1/(z_i^{(0)} - \xi_i)$ (see [8]).

Therefore, if the discs $Z_j^{(0)}$ contain the corresponding zeros ξ_j, $j = 1, \dots, q$, then the disc $Z_i^{(1)}$ on the right-hand side of the last relation is an inclusion disc for the zero ξ_i. This suggests the formulation of the generalised interval method for finding multiple complex zeros of a polynomial.

Using the inclusion (4) we obtain

$$\sum_{\substack{j=1 \\ j \neq i}}^{q} \mu_j \left(\frac{1}{z_i^{(m)} - Z_j^{(m)}}\right)^k \subseteq \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{\mu_j}{(z_i^{(m)} - z_j^{(m)})^k}; \frac{(n - \mu_i)\, k r^{(m)}}{\varrho^{(m)k+1}} \right\} \subseteq$$

$$\subseteq \{c_i^{(m)}; \eta^{(m)}\}. \qquad (6)$$

According to the previous consideration and the inclusion (6), we can formulate the following interval method

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\left\{ \frac{1}{\mu_i} [H_k(z_i^{(m)}) - \{c_i^{(m)}; \eta^{(m)}\}] \right\}_*^{1/k}} \qquad (7)$$

$i = 1, \dots, q; \qquad k \in N; \qquad m = 0, 1, \dots,$

where the symbol * denotes the disc satisfying CCR.

The convergence order of this method is the subject of the next theorem. This theorem asserts that the sequence of radii $(r^{(m)})$ converges to the zero with the convergence order $k + 2$, $k \in N$.

Theorem 2: *Let the interval sequences* $(Z_i^{(m)})$, $i = 1, \dots, q$, *be defined by* (7). *Then, under the condition* (2), *for each* $i = 1, \dots, q$ *and* $m = 0, 1, \dots$

1° $\xi_i \in Z_i^{(m)}$;

2° $r^{(m+1)} < \dfrac{\theta(k, n)\, r^{(m)k+2}}{(\varrho^{(0)} - 3r^{(0)})^{k+1}}, \quad k \in N$.

The center $c_i^{(m)}$, the radius $\eta^{(m)}$ and the constant $\theta(k, n)$ are given in the beginning of this Section.

For the proof of Theorem 2 it is sufficient to replace n by q in all sums and $n-1$ by $n-\mu$ in the proof of Theorem 1 (see the proof mentioned in [8] and the notations in the beginning of this Section). It is easy to show that (2) implies (3) for multiple zeros, too. Hence, the interval process (7) is defined in each iteration, in other words, the disc in the denominator of the formula (7) does not contain the origin so that the inverse disc exists and the formula (1) can be used.

On the basis of the inclusion (6) and of the assertions of Theorem 2, the following theorem, which defines the generalised procedure for multiple complex zeros with error bounds, follows directly.

Theorem 3: *Let the interval sequences* $(Z_i^{(m)})$, $i = 1, \dots, q$, *be defined by*

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\left\{\frac{1}{\mu_i}\left[H_k(z_i^{(m)}) - \sum_{\substack{j=1\\ j\neq i}}^{q} \mu_j \left(\frac{1}{z_i^{(m)} - Z_j^{(m)}}\right)^k\right]\right\}_*^{1/k}} \tag{8}$$

$i = 1, \dots, q;\quad k \in N;\quad m = 0, 1, \dots,$

where the symbol * *denotes the disc which satisfies* CCR.

Then, if (2) *holds, for each* $i = 1, \dots, q$ *and* $m = 0, 1, \dots,$ *the following assertions are valid*

1° $\xi_i \in Z_i^{(m)}$;

2° *the convergence order of the method* (8) *is* $k+2$.

In the special cases, for $k = 1$ and $k = 2$, the interval methods for multiple zeros, given in [2] and [3], follow from (8).

The interval process (8) can be accelerated if we use already calculated inclusion discs in the same iterative step. Let

$$\lambda_j(i) = \begin{cases} m+1 & \text{if } j < i \\ m & \text{if } j \geqq i \end{cases}.$$

Then the accelerated interval method for multiple zeros is given by the formula

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\left\{\frac{1}{\mu_i}\left[H_k(z_i^{(m)}) - \sum_{\substack{j=1\\ j\neq i}}^{q} \mu_j \left(\frac{1}{z_i^{(m)} - Z_j^{(\lambda_j(i))}}\right)^k\right]\right\}_*^{1/k}}, \tag{9}$$

$i = 1, \dots, q;\quad k \in N;\quad m = 0, 1, \dots$

3. Convergence of Interval Centers

Starting from the interval methods, presented in the previous section, we shall formulate the *generalised iterative procedure* (in standard arithmetic) *for the simultaneous finding of multiple complex zeros of a polynomial.*

Assume that the radii $r_i^{(m)}$ of the inclusion discs $Z_i^{(m)}$, $i = 1, \dots, q$, are small enough. Then, from (7) or (8) we obtain the following approximate expression for the center $z_i^{(m+1)}$ of disc $Z_i^{(m+1)}$:

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\left\{\frac{1}{\mu_i}\left[H_k(z_i^{(m)}) - \sum_{\substack{j=1\\ j\neq i}}^{q} \frac{\mu_j}{(z_i^{(m)} - z_j^{(m)})^k}\right]\right\}_*^{1/k}}, \tag{10}$$

$i = 1, \dots, q;\quad k \in N;\quad m = 0, 1, \dots$

The symbol * denotes the choice of one among k values of the k-th root of denominator in (10). A criterion for this selection is given in [8].

The next theorem, whose proof is very simular to the proof of the theorem for simple zeros (see [8]), asserts that the iterative method (10) has the order of convergence equal to the order of the interval methods from which (10) is derived.

Theorem 4: *Let the initial approximations* $z_i^{(0)}$ *of the complex zeros* ξ_i *of the polynomial* P *be chosen so that the iterative sequences* $(z_i^{(m)})$ *converge to* ξ_i, $i = 1, \dots, q$. *Then, the convergence order of the iterative process* (10) *is* $k+2$, $k \in N$.

For example, if $k = 2$, the iterative method of the fourth order

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\left\{\frac{1}{\mu_i}\left[H_2(z_i^{(m)}) - \sum_{\substack{j=1\\ j\neq i}}^{q} \frac{\mu_j}{(z_i^{(m)} - z_j^{(m)})^2}\right]\right\}_*^{1/2}},$$

$i = 1, \dots, q;\quad m = 0, 1, \dots,$

follows from (10). This modification of OSTROWSKI's method (see [4]) provides: (i) increasing the convergence order from three to four; (ii) simultaneous improvement of multiple complex zeros of a polynomial.

Similar to the interval process (8), the iterative procedure (10) can be accelerated, too. From (9) we obtain immediately

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\left\{\frac{1}{\mu_i}\left[H_k(z_i^{(m)}) - \sum_{\substack{j=1\\ j\neq i}}^{q} \frac{\mu_j}{(z_i^{(m)} - z_j^{(\lambda_j(i))})^k}\right]\right\}_*^{1/k}},$$

$i = 1, \dots, q;\quad k \in N;\quad m = 0, 1, \dots$

4. Numerical Results

The presented generalisations of the root iterations were tested on the examples of algebraic equations with multiple complex zeros. These methods were programmed on FORTRAN IV in double precision and realised on the HONEYWELL 66 series.

The polynomial

$$\begin{aligned} P(z) = {} & z^{13} + (-11+4i)\,z^{12} + (46-44i)\,z^{11} + \\ & + (-74+204i)\,z^{10} + (-105-516i)\,z^9 + \\ & + (787+616i)\,z^8 + (-1564+392i)\,z^7 + \\ & + (724-2344i)\,z^6 + (2351+2616i)\,z^5 + \\ & + (-4389+980i)\,z^4 + (430-5148i)\,z^3 + \\ & + (4662+540i)\,z^2 + (-135+2700i)\,z - 675 \end{aligned}$$

was taken to illustrate numerically the interval method (8) for $k = 3$ (the convergence order is five). The factorization of P is

$$P(z) = (z+1)^2\,(z-3)^3\,(z^2-2z+5)^2\,(z+i)^4\,.$$

The initial discs were selected to be $Z_j^{(0)} = \{z_j^{(0)}; 0.5\}$, where $|z_j^{(0)} - \xi_j| \cong 0.35$, $j = 1, \dots, 5$. The distribution of these discs is such that the quotient $\varrho^{(0)}/r^{(0)} = 1.83$ is much less than $k(n-\mu) = 33$. Thus, the choice of initial discs were carried out under weaker condition than (2). But, in spite of large starting regions, after the first iterative step we found that the radii $r_j^{(1)}$ were in the range $[9.68 \times 10^{-3}, 7.25 \times 10^{-4}]$. The following inclusion discs were obtained (Table 1.):

Table 1

j	$Z_j^{(1)}$	μ_j
1	$\{-0.99598 - 1.7 \times 10^{-3}i;\ 9.52 \times 10^{-4}\}$	2
2	$\{3.00011 - 3.52 \times 10^{-5}i;\ 7.25 \times 10^{-4}\}$	3
3	$\{0.99961 + 1.99977i;\ 1.06 \times 10^{-3}\}$	2
4	$\{0.99813 - 2.00387i;\ 9.68 \times 10^{-3}\}$	2
5	$\{-1.25 \times 10^{-4} - 1.00194i;\ 4.57 \times 10^{-3}\}$	4

After the second iterative step, the absolute error, bounded by $r^{(2)}$, was inferior to 9×10^{-13}.

The simultaneous iterative method of the fifth order, defined by (10) for $k = 3$, was applied for finding complex zeros of the polynomial

$$\begin{aligned} P(z) = {} & z^{15} + (6-6i)\,z^{14} + (-12-36i)\,z^{13} + \\ & + (-134+8i)\,z^{12} + (-55+420i)\,z^{11} + \\ & + (1140+330i)\,z^{10} + (1356-2872i)\,z^9 + \\ & + (-5692-4616i)\,z^8 + (-11069+8808i)\,z^7 + \\ & + (12030+21870i)\,z^6 + (39204-12020i)\,z^5 + \\ & + (-2750-61336i)\,z^4 + (-77325-31500i)\,z^3 + \\ & + (-75000+41550i)\,z^2 + (-22500+50000i)\,z + \\ & + 15000i \end{aligned}$$

whose factorization is

$$P(z) = (z - 3)(z + 1)^3 (z - 2i)^3 (z^2 + 4z + 5)^2 (z^2 - 4z + 5)^2 .$$

As the initial approximations of the exact zeros the following complex numbers were taken:

$$z_1^{(0)} = -3.4 + 0.2i, \quad z_2^{(0)} = -0.7 - 0.3i, \quad z_3^{(0)} = 0.3 + 2.4i,$$

$$z_4^{(0)} = -2.3 + 0.6i, \quad z_5^{(0)} = -1.7 - 0.7i, \quad z_6^{(0)} = 2.3 + 1.4i,$$

$$z_7^{(0)} = 1.6 - 0.7i .$$

The results of the first and second iterative step are given in Table 2.

Table 2

j	$z_j^{(1)}$	$z_j^{(2)}$	ν_j
1	$-3.00991 - 6.57 \times 10^{-3}i$	$-3.000000000005241638 - 2.2 \times 10^{-16}i$	1
2	$-0.99751 - 2.5 \times 10^{-3}i$	$-0.999999999999976589 - 5.04 \times 10^{-13}i$	3
3	$-8.53 \times 10^{-4} + 2.00023i$	$3.94 \times 10^{-17} + 2.000000000000000041i$	3
4	$-1.98889 + 1.01753i$	$-2.000000000037126743 + 1.000000000043274417i$	2
5	$-2.02133 - 0.99856i$	$-2.000000000035192840 - 1.000000000524641291i$	2
6	$1.99751 + 0.99922i$	$2.000000000000000554 + 0.999999990099997861i$	2
7	$1.99955 - 0.99829i$	$2.000000000000000112 - 0.999999990099990801i$	2

The greatest distances of the exact zeros

$$\varepsilon^{(m)} = \max_{1 \le j \le q} |z_j^{(m)} - \xi_j|, \qquad m = 0, 1, \ldots,$$

are $\varepsilon^{(1)} \cong 2.1 \times 10^{-2}$ and $\varepsilon^{(2)} \cong 6.32 \times 10^{-10}$.

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Differential Systems and Multipoint Boundary Value Problems

Let $I = [a, b]$ be a compact real interval with $a = x_1 < \ldots < x_k = b$. Let $D(x)$ be an $n \times n$ matrix with components in $C(I)$ and let $M_1, \ldots, M_k$ be $n \times n$ matrices. A Green's matrix for the multipoint boundary value problem for the n-dimensional system $y' - D(x)\,y = 0$, $Ty \equiv \sum_{i=1}^{k} M_i y(x_i) = 0$, is constructed; inclusion theorems for the problem $y' = f(x, y)$, $Ty = \gamma$, where $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $\gamma \in \mathbb{R}^n$, are obtained. The inclusion theorems depend upon partial orderings which are constructed according to the Green's matrix.

1. Introduction

Many investigators have studied boundary value problems for higher order differential equations. The two-point conjugate type problem has perhaps received the most attention, but the theory of focal type problems and various multipoint problems has undergone rapid development. The sign properties of the Green's function, established by LEVIN [1] for multipoint conjugate type problems and some analogous results for the Green's functions and their derivatives, obtained recently by PETERSON [4] and ELIAS [2] have been valuable in the connection. However, such conditions are not always available, and it is of interest to study problems without using explicit sign conditions on the Green's function, as well as to obtain results for systems.

Some time ago, WERNER [5] obtained inclusion theorems for two-point boundary value problems for first order differential systems, by using a method of induced monotonicity. In this paper, we show how these results can be extended to various multipoint problems. Results which are somewhat related, but require explicit use of the Green's function sign properties were obtained for nth order scalar equations in our earlier paper [3].

Let $n \geqq 1$, let $I = [a, b]$ be a real interval, and let $a = x_1 < x_2 < \ldots < x_k = b$. Let $M_1, M_2, \ldots, M_k$ be $n \times n$ constant matrices and let $\gamma \in \mathbb{R}^n$. Let $C_n(I)$ be the set of continuous n-vector-valued functions on I and define the linear operator $T\colon C_n(I) \to \mathbb{R}^n$ by

$$Ty = \sum_{j=1}^{k} M_j y(x_j) . \tag{1}$$

Consider the k-point boundary value problem (BVP) for the n-dimensional system,

$$y' = f(x, y) , \quad Ty = \gamma , \tag{2) (3}$$

where $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

We shall establish existence results for solutions of (2) – (3) based on solutions of certain differential inequalities; in addition, we present iteration schemes that approximate solutions of (2) – (3).

Let $D(x)$ be an $n \times n$ matrix with elements in $C(I)$ and consider an equivalent system,

$$y' - D(x)\,y = f(x, y) - D(x)\,y , \qquad Ty = \gamma . \tag{4) (5}$$

If $y \equiv 0$ is the only solution of the associated homogeneous problem,

$$y' - D(x)\,y = 0 , \qquad Ty = 0 , \tag{6) (7}$$

WERNER [5] has proved existence theorems for solutions of (4) – (5) and hence, of (2) – (3), in the case where $k = 2$. We shall construct a Green's matrix $G(x, s)$ for (6) – (7), which we can employ to extend WERNER's results to the multipoint case.

2. Definitions and preliminary results

Consider the Banach space $C_n(I)$ with the norm

$$\|y\| = \max_{i=1,\ldots,n} \max_{x \in I} |y_i(x)| .$$

We define several *partial orderings* in $C_n(I)$.

i) For $y, z \in C_n(I)$, define the relation '$\leqq$' by $y \leqq z$ if and only if $y_i(x) \leqq z_i(x)$, for $x \in I$, $i = 1, \ldots, n$. This relation is a partial ordering in $C_n(I)$, which we shall call the *natural partial ordering* in $C_n(I)$. It can be shown that $C_n(I)$ is a partially ordered Banach space with respect to the natural partial ordering.

ii) Let $H\colon C_n(I) \to C_n(I)$ be an invertible linear operator. For $y, z \in C_n(I)$, define the relation '$\leqq_H$' by

$$y \leqq_H z \text{ if and only if } Hy \leqq Hz .$$

The relation '$\leqq_H$' is a partial ordering in $C_n(I)$ and we say that '$\leqq_H$' is the *partial ordering induced by H*. Again, it can be