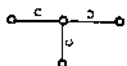


## RECTAGRAPHS, DIAGRAMS, AND SUZUKI'S SPORADIC SIMPLE GROUP

A. NEUMAIER

*Dedicated to N.S. Mendelsohn on the occasion of his 65th birthday*

We investigate incidence structures with diagram  $\begin{array}{ccccccc} \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\ \parallel & \parallel & & \parallel & \parallel & \parallel & \parallel \\ 2 & 2 & & 2 & 2 & 2 & k-1 \end{array}$  of rank  $n+1$ . For  $n=2$ , examples can be constructed from biplanes, semibiplanes, and strongly regular graphs with  $\lambda=0$ ,  $\mu=2$ . For  $n>2$ , there are some examples related to the construction of a strongly regular graph on 1782 points by Suzuki. We also give some general theorems on diagrams which, e.g., imply that an incidence structure with diagram



gives rise to another incidence structure with diagram  $\begin{array}{ccccccc} \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\ \parallel & \parallel & & \parallel & \parallel & \parallel & \parallel \\ 2 & 2 & & 2 & 2 & 2 & c \end{array}$ ; an example of this situation can be obtained from the rank 4 representation of  $PSU_3(9)$  on 36 points.

### 1. Rectagraphs

All our graphs are finite, undirected, without loops or multiple edges. The *neighbourhood* of a set  $C$  of vertices (of a vertex  $x$ ) of a graph  $\Gamma$  is the set  $\Gamma(C)$  (resp.  $\Gamma(x)$ ) of all vertices adjacent with all vertices of  $C$  (resp. with  $x$ ).  $\Gamma$  is *regular of valency  $k$*  if every vertex is adjacent with exactly  $k$  other vertices. Often a vertex is simply called a *point*. An *s-claw* is a pair  $(x, S)$  where  $x$  is a point, and  $S$  is a set of  $s$  points adjacent with  $x$  such that no two points of  $S$  are adjacent. An *n-clique* is a set of  $n$  pairwise adjacent points.

A *rectagraph* (cf. [8]) is a connected, triangle-free graph with the property that each 2-claw is in a unique quadrangle. Perkel [8] shows that a rectagraph is always regular.

**Examples.** (1) Strongly regular graphs with parameters  $(\lambda=0, \mu=2)$  are rectagraphs. By well-known parameter conditions (see, e.g., [7]) the valency must be of the form  $k = s^2 + 1$ , with an integer  $s$  not divisible by 4, and the number of vertices is then  $v = \frac{1}{2}(s^2 + s + 2)(s^2 - s + 2)$ . Examples are known for  $s=1$  (quadrangle,  $k=2, v=4$ ), for  $s=2$  (Clebsch graph,  $k=5, v=16$ ) and for  $s=3$  (Gewirtz graph,  $k=10, v=56$ ) (see, e.g., [7]). By considering the set of points at

distance  $\leq 2$  from a given point it is easy to see that a rectagraph  $\Gamma$  of valency  $k$  contains  $v \geq 1 + k + \binom{k}{2}$  points, with equality iff  $\Gamma$  has diameter 2 iff  $\Gamma$  is a strongly regular graph with parameters  $\lambda = 0, \mu = 2$ . These graphs also arise in connection with algebraic varieties in projective spaces (see [1]).

(2) The  $k$ -dimensional cubes (and, for  $k \geq 5$ , the half-cubes obtained by identifying antipodal vertices) are rectagraphs of valency  $k$ .

(3) A *biplane* (*semibiplane*) is a design with the property that any two distinct points are in 2 (0 or 2) blocks, and any two distinct blocks have 2 (0 or 2) common points. The incidence graph of a biplane or semibiplane  $\mathcal{B}$ , i.e., the graph whose vertices are the points and blocks of  $\mathcal{B}$ , adjacent iff they are incident, is a rectagraph whose valency equals the block size. It is easy to see that a rectagraph is bipartite iff it is the incidence graph of a semibiplane. Also, a bipartite rectagraph comes from a biplane iff its diameter is 3.

**Proposition 1.** *In a rectagraph, if  $x, y$  are points at distance  $i$ , then there are at least  $i$  points adjacent to  $y$  which have distance  $i - 1$  from  $x$ .*

**Proof.** This is true for  $i = 1$ , so assume that it is true for  $i = 1, \dots, j$ . If  $x, y$  are at distance  $j + 1$  then there is a chain  $x = x_0, x_1, \dots, x_{j+1} = y$  of adjacent points. By induction, there are at least  $j$  points  $s_1, \dots, s_j$  adjacent to  $x_j$  which have distance  $j - 1$  from  $x$ .

Now define  $t_0 = x_j$  and define  $t_l$  as the fourth point on the quadrangle containing the 2-claw  $s_l x_j x_{j+1}$  ( $l = 1, \dots, j$ ). Then  $t_0, t_1, \dots, t_j$  are  $j + 1$  distinct points adjacent to  $x_{j+1}$  and at distance  $j$  from  $x$ . This completes the induction.

**Proposition 2.** *A rectagraph  $\Gamma$  of valency  $k$  has diameter  $\leq k$ , and contains  $v \leq 2^k$  points. Moreover,  $\Gamma$  has diameter  $k$  iff  $v = 2^k$  iff  $\Gamma$  is a  $k$ -dimensional cube.*

**Proof.** By Proposition 1, the diameter is  $\leq k$ . If  $v_i$  is the number of points at distance  $i$  from a given vertex  $x$  then  $v_0 = 1$ . If we count in two ways the number  $N_i$  of edges  $yz$  with  $y$  at distance  $i$ ,  $z$  at distance  $i + 1$  from  $x$ , we obtain  $v_{i+1}(i + 1) \leq N_i \leq v_i(k - i)$ . Hence, by induction,  $v_i \leq \binom{k}{i}$ , and  $v \leq \sum \binom{k}{i} = 2^k$ .

Now suppose that the diameter of  $\Gamma$  is  $k$ . Choose two points  $a, b$  at distance  $k$ , and define  $\Gamma_0$  as the set of points whose distances to  $a$  and  $b$  sum up to  $k$ . If  $x \in \Gamma_0$  has distance  $i$  from  $a$  and  $k - i$  from  $b$  then by Proposition 1, there are at least  $i$  vertices at distance  $i - 1$  from  $a$ , and at least  $k - i$  vertices at distance  $k - i - 1$  from  $b$ , hence at distance  $i + 1$  from  $a$ , which are adjacent with  $x$ . But the valency is  $k$ , whence there are exactly  $i$  vertices at distance  $i - 1$ , and  $k - i$  vertices at distance  $i + 1$  from  $a$  in the neighbourhood of  $x$ , and so  $\Gamma(x)$  is in  $\Gamma_0$ . Since  $a \in \Gamma_0$  and  $\Gamma$  is connected we now have  $\Gamma_0 = \Gamma$ , and the above counting argument gives  $v_{i+1}(i + 1) = N_i = v_i(k - i)$ , or  $v_i = \binom{k}{i}$ . Hence  $v = 2^k$ .

Finally, if  $v = 2^k$  then  $v_i = \binom{k}{i}$  for all  $i$ , in particular  $v_k = 1$ . Hence for every point  $a$  there is a point at distance  $k$  from  $a$ . In particular, by the above arguments, Proposition 1 holds with 'at least' replaced by 'exactly'. Now fix  $a \in \Gamma$ , and identify  $x \in \Gamma$  (at distance  $i$  from  $a$ ) with the characteristic vector  $c(x)$  of the set of all  $y \in \Gamma(a)$  at distance  $i - 1$  from  $x$ . Then  $c(x)$  has  $i$  entries, and it is easy to show that  $c$  is an isomorphism from  $\Gamma$  into the  $k$ -dimensional cube.

**Remark.** This result has been obtained for the incidence graph of a semi-biplane by Wild [10] with a similar proof. Wild also shows that a bipartite rectagraph  $\Gamma$  of valency  $k$  contains  $v \geq k^2 - k + 2$  vertices, with equality iff  $\Gamma$  is the incidence graph of a biplane.

**Proposition 3.** *Let  $\Gamma$  be a rectagraph on  $X$  with valency  $k$ . Then the graph  $\Gamma_\varepsilon$  whose vertices are the pairs  $(x, \alpha)$  with  $x \in X$ ,  $\alpha = \pm 1$ , such that  $(x, \alpha)$  and  $(x', \alpha')$  are adjacent iff either  $x = x'$  and  $\alpha\alpha' = -1$ , or  $x'$  is adjacent with  $x$  and  $\alpha\alpha' = \varepsilon$ , is a rectagraph of valency  $k + 1$ , for  $\varepsilon = \pm 1$ .*

**Proof.** Straightforward.

**Remarks.** (1) If  $\Gamma$  is bipartite, with classes  $C^\alpha$ ,  $\alpha = \pm 1$ , then  $\Gamma_+$  is bipartite, with classes  $\{(x, \alpha) \mid x \in C^\alpha, \alpha = \pm 1\}$ ,  $\{(x, -\alpha) \mid x \in C^\alpha, \alpha = \pm 1\}$ . Hence this construction generalizes Wild's doubling construction [10] for semiplanes. If  $\Gamma$  is a  $k$ -dimensional cube then  $\Gamma_+$  is a  $(k + 1)$ -dimensional cube.

(2)  $\Gamma_-$  is always bipartite, with classes  $\{(x, \alpha) \mid x \in X\}$  for  $\alpha = \pm 1$ . If  $\Gamma$  is strongly regular, then  $\Gamma_-$  is the incidence graph of a biplane with a null polarity. This is a special case of a construction of symmetric 2-designs with a null polarity from strongly regular graphs with  $\lambda = \mu - 2$  (see, e.g., [7]). If  $\Gamma$  is a  $k$ -dimensional cube, then, again,  $\Gamma_-$  is a  $(k + 1)$ -dimensional cube. If  $\Gamma$  is a half-cube, then  $\Gamma_-$  is a cube with additional edges joining antipodal pairs of points.

(3) In the graph  $\Gamma'_-$  obtained from  $\Gamma_-$  by deleting the edges  $(x, 1)$ ,  $(x, -1)$ , each 2-claw is in a unique quadrangle.  $\Gamma'_-$  is bipartite, and, unless  $\Gamma$  itself is bipartite, connected, whence a rectagraph. In particular, every rectagraph is contained in some bipartite rectagraph of the same valency, and a rectagraph of valency  $k$  which is not bipartite has at most  $2^{k+1}$  vertices.

## 2. Diagrams

For the remainder, we assume the reader to be familiar with the paper "Diagrams for geometries and groups" by Buekenhout [2]. We take a slightly



then there also is a geometry with diagram



This geometry has an  $s$ -partite point graph, and conversely, if a geometry with diagram (2) has an  $s$ -partite point graph, then there is a geometry with diagram (1).

(ii) If there is a geometry with diagram

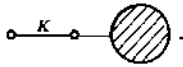


then there also is a geometry with diagram



This geometry has a bipartite point graph, and conversely, if a geometry with diagram (4) has a bipartite point graph, then there is a geometry with diagram (3).

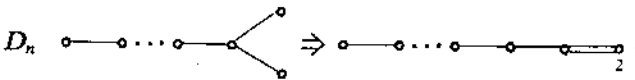
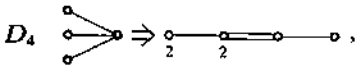
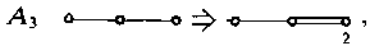
**Proof.** (i) Call the varieties at the  $s$  distinguished nodes points more specifically 1-points, ...,  $s$ -points. Replace these varieties by new varieties which we call 1-spaces, ...,  $s$ -spaces. An  $i$ -space is simply a flag consisting of  $i$  distinct points. Incidence between  $i$ -spaces is containment, and incidence between an  $i$ -space and an old variety is as before. The residue of an  $(s - 1)$ -space has diagram



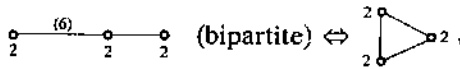
The residue of a variety belonging to the central node consists of the cliques of a complete  $s$ -partite graph, hence has diagram  $\overset{2}{\circ} - \overset{2}{\circ} \cdots \overset{2}{\circ} - \overset{2}{\circ}$ . Hence the new geometry has diagram (2). The converse (in the form stated) is straightforward.

(ii) is proved in the same way, using the observation that the incidence graph of a generalized  $n$ -gon is a generalized  $2n$ -gon with line size 2.

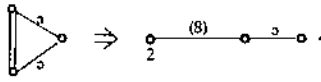
Theorem 4 contains various classical examples as special cases:



It also applies to the (bipartite) tessellation of a plane by regular hexagons:



and to a diagram for Held's group (see [11]):



Here are two more constructions, the first of which generalizes Proposition 3.

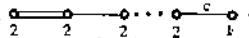
**Proposition 5.** (i) Every geometry with diagram



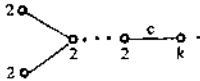
gives rise to a geometry with the same diagram and bipartite point graph, and hence to a geometry with diagram



(ii) If there is a geometry with diagram (5), then there is a geometry of the same rank with diagram



(iii) If there is a geometry with diagram (6), then there is a geometry of the same rank with diagram



**Proof.** (i) Let  $S$  be a geometry with diagram (5) whose point graph is not bipartite. Take as new point set a red and a blue copy of the old point set, and for each  $i$ -variety ( $i > 0$ ), which is an  $i$ -dimensional cube on the incident points, define two new  $i$ -varieties corresponding to a red-blue and a blue-red colouring of the natural bipartition of the cube. This defines naturally a new geometry with diagram (5). If we separate the red points and the blue points, then, by Theorem 4, we get a geometry with diagram (6).

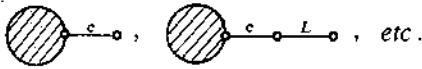
(ii) Define new points  $(x_0, \alpha)$ , where  $x_0$  is an old point,  $\alpha = \pm 1$ . For  $i > 0$ , define new  $i$ -varieties to be either  $(x_i, \alpha)$ , where  $x_i$  is an old  $i$ -variety,  $\alpha = \pm 1$ , or  $(x_{i-1})$ , where  $x_{i-1}$  is an old  $(i-1)$ -variety. The incidence is given by  $(x_i, \alpha) \text{ I } (x_j, \alpha')$  iff  $x_i \text{ I } x_j, \alpha = \alpha'$ ;  $(x_i, \alpha) \text{ I } (x_j)$  iff  $x_i \text{ I } x_j, i \leq j$ ;  $(x_i) \text{ I } (x_j)$  iff  $x_i \text{ I } x_j$ . The verification of the diagram is straightforward.

(iii) follows from (ii) by Remark 2 of Section 1.

**Proposition 6.** *If there is a geometry with diagram*

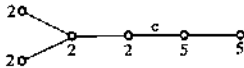


*then there also are geometries of smaller rank with diagrams*

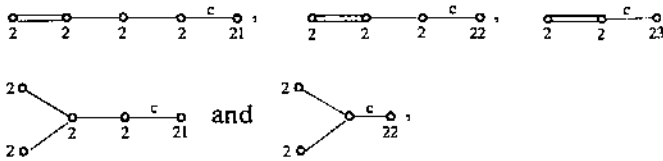


**Proof.** Delete the varieties belonging to the deleted nodes of diagram (7) and apply [2, Theorem 7].

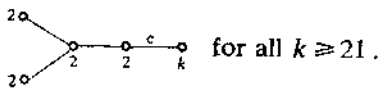
**Example.** By Cameron [6] the group  $2^{12}M_{24}$  acts on the  $2^{12}$  cosets of the binary Golay code as the graph of a parallelism of the 4-subsets of a 24-set. This gives rise to a geometry with diagram  $\frac{2}{2} - \frac{2}{2} - \frac{2}{2} - \frac{c}{2} - \frac{5}{5} - \frac{5}{5}$ ; the varieties are points, edges, quadrangles, 3-cubes, 4-cubes and 8-cubes corresponding to certain 0-, 1-, 2-, 3-, 4-, and 8-dimensional subspaces. Moreover, the point graph is bipartite, whence there is a geometry for the diagram



(the group is  $2^{11}M_{24}$ , and the diagram was mentioned in [2, Example 12.3]). Truncation according to Proposition 6 gives geometries for the diagrams



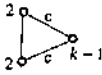
and then Proposition 5 shows the existence of geometries with diagram



Finally we mention that the incidence graph of a thick partial plane with the property that any two points on a line are on a unique triangle give rise to a geometry with diagram  $\frac{2}{2} - \frac{(6)}{2} - \frac{c}{2}$ , hence to a geometry with diagram



If every point is in  $k$  lines and every line contains  $k$  points, then we have the diagrams



and  $\overset{(6)}{\circ} \text{---} \overset{c}{\circ} \text{---} \overset{c}{\circ}$ . Unfortunately, I don't know any example of this situation.

### 3. Geometries of type $S_n$

For the purpose of the next theorem, call a group  $H$  acting on a set  $X$  fully 2-transitive if it is 2-transitive and the pointwise stabilizer of two points fixes no other point.

**Theorem 7.** Let  $\Gamma_n$  be a graph with the property that, for some  $n \geq 2$ , (i) there are  $n$ -cliques, and (ii) the neighbourhood of a  $(n-2)$ -clique is a rectagraph of valency  $k$ .

Suppose that  $\Gamma_n$  possesses a group  $G_n$  of automorphisms such that (iii)  $G_n$  is transitive on  $n$ -cliques, and the stabilizer of an  $n$ -clique induces on it the symmetric group  $S_n$ , and (iv) the stabilizer  $H$  of a  $(n-1)$ -clique  $C$  is fully 2-transitive on the neighbourhood of  $C$ .

If  $\Gamma_n$  is connected, then it is the point graph of a geometry  $S$  of rank  $n+1$  with diagram



and  $G_n$  is transitive on the maximal flags of  $S$ .

**Proof.** Let us call an  $n$ -clique  $B$  of  $\Gamma_n$  a mate of another  $n$ -clique  $C$  of  $\Gamma_n$  if  $B \cup C$  is a  $K_{n \times 2}$ , a complete multipartite graph with  $n$  classes of size 2. Let  $K$  be a group conjugate to a two-point stabilizer of  $H$ , and denote by  $F_n(K)$  the fixpoint set of  $K$  in  $\Gamma_n$ . To facilitate the induction we also introduce the empty graph  $\Gamma_0$ , and the graph  $\Gamma_1$  with  $k$  vertices and no edge. If  $\Gamma_m, G_m$  satisfy the hypothesis of the theorem, then so do the neighbourhood  $\Gamma_{m-1} = \Gamma_m(x)$  together with the stabilizer  $G_{m-1} = (G_m)_x$  of any point  $x$ , with the same group  $H$ . Hence the neighbourhood in  $F_m(K)$  of a point of  $F_m(K)$  is a  $F_{m-1}(K)$ . The essential argument is the following lemma.

**Lemma 8.** For all integers  $n \geq 0$ , every  $n$ -clique contained in  $F_n(K)$  has exactly one mate in  $F_n(K)$ .



**Proof.** The lemma is trivial for  $n = 0$  and true for  $n = 1$  since, by (iv),  $F_1(K)$  contains exactly two (nonadjacent) points. For  $n = 2$ , if  $F_2(K)$  contains an edge  $xy$ , then it contains, by definition of  $K$ , a 2-claw  $(x, \{yy'\})$  which is in a quadrangle  $xyx'y'$ , and  $x'y'$  is a mate of  $xy$  ( $x'$  is fixed by  $K$  since it is uniquely determined by  $x, y, y'$ ). If  $x''y''$  is another mate of  $xy$  then  $x'$  and  $x''$  are mates of  $x$  in the neighbourhood of  $y$ , hence  $x' = x''$ , and similarly  $y' = y''$ . Hence the mate is unique. Now we proceed by induction, and assume Lemma 8 for  $0, 1, \dots, n-1$  in place of  $n$ .

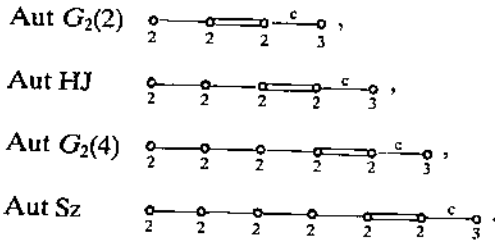
For  $n \geq 3$ , if  $F_n(K)$  contains an  $n$ -clique  $C$ , choose three distinct points  $x_i \in C$ ,  $i = 1, 2, 3$ . Let  $B$  be the mate of  $C' = C - \{x_1, x_2, x_3\}$  in the neighbourhood  $F_{n-3}(K)$  of  $x_1, x_2, x_3$ . Let  $B_i$  be the mate of  $C - \{x_i\}$  in the neighbourhood of  $x_i$  ( $i = 1, 2, 3$ ), and let  $x_{ji}$  be the opposite of  $x_j$  in  $B_i$  ( $j \neq i$ ). Then  $B_1 - \{x_{21}, x_{31}\}$  is a mate of  $C'$  in  $F_{n-3}(K)$  whence  $B_1 = B \cup \{x_{21}, x_{31}\}$ , and similarly  $B_2 = B \cup \{x_{32}, x_{12}\}$ ,  $B_3 = B \cup \{x_{13}, x_{23}\}$ . Therefore,  $B \cup \{x_{12}\}$  and  $B \cup \{x_{13}\}$  are mates of  $C - \{x_2, x_3\}$  in the neighbourhood of  $x_2$  and  $x_3$ , whence  $x_{12} = x_{13} = y_1$ , say, and similarly  $x_{23} = x_{21} = y_2$ ,  $x_{31} = x_{32} = y_3$ . Now it is easy to see that  $B \cup \{y_1, y_2, y_3\}$  is an  $n$ -clique, hence a mate of  $C$ , and that this mate is unique.

**Proof of Theorem 7 (continued).** Let  $C$  and  $C'$  be  $n$ -cliques intersecting in a  $(n-1)$ -clique  $C_0$ , so that  $C = C_0 \cup \{a\}$ ,  $C' = C_0 \cup \{b\}$ , with two nonadjacent points  $a, b$  in the neighbourhood of  $C_0$ . By (iv) the pointwise stabilizer  $K$  of  $C \cup C'$  is conjugate to a two-point stabilizer of  $H$ . By Lemma 8,  $C$  has a unique mate  $B$  in  $F_n(K)$ . For  $x \in C$ , denote by  $x'$  the opposite of  $x$  in  $B \cup C$ . An application of Lemma 8 to  $\Gamma_1 = \Gamma_n(C_0)$ , with  $n = 1$ , shows that  $b = a'$  whence  $B$  contains  $b$ . Now let  $\bar{B}$  be any mate of  $C$  in  $\Gamma_n$  containing  $b$ . For  $x \in C$ , denote by  $\bar{x}$  the opposite of  $x$  in  $\bar{B} \cup C$ . Then  $\bar{a} = b = a'$ . If  $x \in C - \{a\}$ , then in the neighbourhood of the  $(n-2)$ -clique  $C - \{a, x\}$ ,  $x\bar{x}a'$  and  $xax'a'$  are quadrangles containing the 2-claw  $(x, \{aa'\})$ , whence by (ii),  $\bar{x} = x'$ . Hence  $\bar{B} = B$ , and  $C \cup C'$  is in the unique  $K_{n \times 2}$   $B \cup C$ . Hence we have the following:

(P<sub>n</sub>) Two  $n$ -cliques which intersect in a  $(n-1)$ -clique are in a unique  $K_{n \times 2}$ .

Now it is easy to check that we obtain a geometry  $S$  of rank  $n+1$  with the required properties, if we call the  $(i+1)$ -cliques  $i$ -varieties (for  $i = 0, \dots, n-1$ ), and the  $K_{n \times 2}$   $n$ -varieties.

**Theorem 9.** *There are geometries with the following groups and diagrams:*

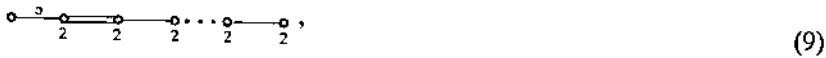


**Proof.** Suzuki [9] constructs strongly regular graphs  $\Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$  with 36, 100, 416, and 1782 points and transitive automorphism groups  $G_3 = \text{Aut } G_2(2)$ ,  $G_4 = \text{Aut HJ}$ ,  $G_5 = \text{Aut } G_2(4)$ , and  $G_6 = \text{Aut Sz}$ , such that the neighbourhood (stabilizer) of a point of  $\Gamma_i$  is isomorphic to  $\Gamma_{i-1}$  ( $G_{i-1}$ ), for  $i = 6, 5, 4, 3$ ; here  $\Gamma_2$  is a rectagraph with 14 points, namely the incidence graph of the unique biplane with  $k = 4$ , and  $G_2 = \text{Aut PSL}_3(2)$ . Hence these graphs satisfy the conditions of Theorem 7 with  $k = 4$ ,  $H = S_4$ .

Now we consider, in a slightly more general setting, the dual geometries. Let  $\Gamma_n$  be a graph satisfying  $(P_n)$  and let the following hold:

- $(Q_n)$  Every maximal clique has  $n$  points, and every nonmaximal clique is in at least 3 maximal cliques.

Define a geometry  $S$  by calling the  $K_{n \times 2}$  points (0-varieties) and the  $i$ -cliques  $(n + 1 - i)$ -varieties ( $i = 1, \dots, n$ ), with natural incidence. This gives a geometry of rank  $n + 1$  with diagram



and  $\Gamma_n$  can be recovered as the graph on the blocks ( $n$ -varieties) defined by calling two blocks adjacent if they are distinct and incident with a common line (1-variety). A geometry arising in this way from a graph  $\Gamma_n$  with  $(P_n)$  and  $(Q_n)$  is said to be of type  $S_n$ . It can be shown that a geometry with diagram (9) is of type  $S_n$  iff the block graph satisfies  $(P_n)$  and  $(Q_n)$ . On the other hand, the dual of the half-cube [2, p. 128] has rank 3 and diagram (9) with 3 nodes but a complete block graph, hence is not of type  $S_n$ .

Note that the residue of a block in a geometry of type  $S_n$  is a geometry of type  $S_{n-1}$ .

Let  $S$  be a geometry of type  $S_n$ . For a variety  $v \in S$ , denote by  $P(v)$ ,  $B(v)$  the set of points resp. blocks of  $S$  incident with  $v$ . Then  $B(v)$  is a  $K_{n \times 2}$  if  $v$  is a point, an  $(n + 1 - i)$ -clique if  $v$  is an  $i$ -variety,  $i > 0$ , and for every  $i$ -clique  $B$  of blocks there is a unique  $(n + 1 - i)$ -variety  $v$  with  $B(V) = B$ , and every  $K_{n \times 2}$  is of the form  $B(x)$  with a point  $x$ .

**Proposition 10.** Let  $S$  be a geometry of type  $S_n$ , and let  $v, w$  be varieties of  $S$ . Then

- (i)  $P(v) \subseteq P(w)$  iff  $v \leq w$  iff  $B(v) \supseteq B(w)$ ,  
 (ii)  $P(v) = P(w)$  iff  $v = w$  iff  $B(v) = B(w)$ .

**Proof.** Obviously,  $v \leq w$  iff  $B(v) \supseteq B(w)$ , and  $v \leq w$  implies  $P(v) \subseteq P(w)$ . Assume that  $v \not\leq w$  and  $v$  is not a point. Then  $B(v)$  is a clique not containing  $B(w)$ . Hence there is a block  $b \in B(w)$  with  $b \notin B(v)$ . By  $(Q_n)$  there is a  $n$ -clique  $B \supseteq B(v)$  with  $b \notin B$ . By  $(P_n)$  and  $(Q_n)$  there are at least 3  $K_{n \times 2}$  containing  $B$ , but at most one containing  $B$  and  $b$ . Hence there is a  $K_{n \times 2}$  containing  $B$  but not  $b$ , and a corresponding point  $x \in S$  such that  $b \notin B(x) \supseteq B$ . Then  $B(v) \subseteq B(x)$ ,  $B(w) \not\subseteq B(x)$ , whence  $x \leq v$  but  $x \not\leq w$ . Hence  $P(v) \not\subseteq P(w)$ , which proves (i). (ii) is a consequence of (i).

By Proposition 10 we may identify a variety  $v$  with the point set  $P(v)$ , and we may talk about the intersection of varieties. For the next proposition, call two points *adjacent* if they are on a line, and call a *triangle* a set of 3 mutually adjacent points not contained in a line.

**Proposition 11.** In a geometry of type  $S_n$ , any two adjacent points are on a unique line, and every triangle is in a unique plane.

**Proof.** Let  $l, l'$  be two distinct intersecting lines. Then  $B(l), B(l')$  are distinct  $n$ -subcliques of a  $K_{n \times 2}$ , and there are nonadjacent blocks  $b \in B(l), b' \in B(l')$ . Let  $\pi$  be the plane with  $B(\pi) = B(l) - \{b\}$ , and let  $l''$  be the line with  $B(l'') = B(\pi) \cup \{b'\}$ . Then  $B(\pi) \subseteq B(l) \cap B(l'')$  whence  $l$  and  $l''$  are in  $\pi$ . Since  $\pi$  is a  $\circ \text{---} \circ \text{---} \circ$ ,  $l$  and  $l''$  intersect in a unique point. Now if  $x$  is a point incident with  $l$  and  $l'$  then  $B(x) \supseteq B(l) \cup B(l') \supseteq B(l'')$  whence  $x$  is also incident with  $l''$ . Hence  $l$  and  $l'$  have a unique point in common, and so two adjacent points  $xy$  are on a unique line  $\overline{xy}$ .

Now let  $xyz$  be a triangle not in a block. Define  $B_1 = B(\overline{xy}), B_2 = B(\overline{xz}), B_3 = B(\overline{yz})$ . Then  $B_1, B_2, B_3$  are disjoint  $n$ -cliques. Since  $B(x) \supseteq B_1 \cup B_2$ , and contains  $2n$  blocks,  $B(x) = B_1 \cup B_2$ , and similarly  $B(y) = B_1 \cup B_3, B(z) = B_2 \cup B_3$ . Now  $B(x), B(y), B(z)$  are  $K_{n \times 2}$  whence  $B = B_1 \cup B_2 \cup B_3$  contains  $3n$  blocks, and every block is nonadjacent to exactly two other blocks. If  $B$  contains a triple  $(b_0, b_1, b_2)$  such that  $b_0$  is not adjacent with  $b_1$  and  $b_2$ , but  $b_1$  and  $b_2$  are adjacent, then  $C_2 = \{b_1, b_2\}$  may be extended to a 3-clique  $C_3$  in at least  $3n - 5$  ways,  $C_3$  to a 4-clique  $C_4$  in at least  $3n - 8$  ways, etc., and we see that there is a  $(n + 1)$ -clique, violating  $(Q_n)$ . Hence, if  $b_0$  is nonadjacent with  $b_1$  and  $b_2$ , then  $b_1$  and  $b_2$  are nonadjacent. Therefore,  $B$  is a  $K_{n \times 3}$ . But this contradicts  $(P_n)$ . Hence every triangle is in a block. Repetition of the argument in the residue of this block, etc.,

shows now that every triangle is in a plane. By part (i) of the next proposition, this plane is unique.

**Proposition 12.** *Let  $S$  be a geometry of type  $S_m$ , and suppose that, for some  $m \leq n$ , if two varieties are contained in some  $m$ -variety, then their intersection is a variety or empty. (This holds, e.g., for  $m = 2$ .) Then the following holds:*

- (i) *Two varieties containing a common line intersect in a variety.*
- (ii) *A variety intersects an  $i$ -variety with  $i < m$  in a subspace or empty.*

**Proof.** The hypothesis holds for  $m = 2$  since every plane is a  $\circ \text{---}^2 \text{---} \circ$ , hence a partial plane.

(i) Let  $v, w$  be varieties containing the line  $l$ . Then  $B(v) \cup B(w)$  is a subset of  $B(l)$ , hence a clique. Therefore, there is a variety  $u$  such that  $B(v) \cup B(w) = B(u)$ , and a point  $x$  is in  $v \cap w$  iff  $B(x) \supseteq B(v) \cup B(w) = B(u)$  iff  $x$  is in  $u$ . Hence  $v \cap w = u$ .

(ii) Let  $v, w$  be varieties such that  $v \cap w$  is neither empty nor a variety. We may choose  $v, w$  such that the rank of  $w$  is minimal, say,  $w$  is an  $i$ -variety. Then neither of  $v, w$  is a point, and by (i),  $v$  and  $w$  have no common line. Hence  $B(v), B(w)$  are cliques, and  $B(v) \cup B(w)$  is not a clique. So there are nonadjacent blocks  $b \in B(v), c \in B(w)$ . Let  $u$  be the  $(i+1)$ -variety with  $B(u) = B(w) - \{c\}$ , and let  $v'$  be the  $i$ -variety with  $B(v') = B(u) \cup \{b\}$ . If  $x \in v \cap w$ , then  $B(x) \supseteq B(v) \cup B(w) \supseteq B(v') \cup B(w)$ , whence  $x \in v' \cap w$ . Therefore  $v \cap w \subseteq v' \cap w$ . If  $v' \cap w = w'$  is a variety, then  $v \cap w' \subseteq v \cap w = v \cap (v \cap w) \subseteq v \cap (v' \cap w) = v \cap w'$  and so  $v \cap w' = v \cap w$  is not a variety. By the minimality of the rank of  $w, w' = w$ , i.e.,  $w \leq v'$  contrary to our construction. Hence  $v' \cap w$  is not a variety. But  $v'$  and  $w$  are contained in the  $(i+1)$ -variety  $u$ , hence, for  $i < m$ , in some  $m$ -variety, contradiction.

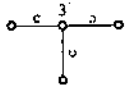
**Theorem 13.** *For  $n = 3, 4, 5, 6$ , there are geometries with diagram  $\circ \text{---}^3 \text{---} \circ \text{---}^2 \text{---} \circ \text{---}^c \text{---} \circ$  satisfying axiom (3) of [2]. The lines have 3 points, and if  $abc, ade, bef$  are lines, then  $cdf$  is a line.*

**Proof.** By [2, Example 12.1 and Theorem 6] the duals of the geometries of Theorem 9 satisfy the assumptions of Proposition 12 with  $m = 4$ . Now apply Proposition 6 to get the diagram, [2, Theorem 6] to get axiom (3), and Proposition 11 to get the closure property for the lines.

**Remarks.** (1) In case of rank  $n+1 \geq 5$ , the geometries of Theorem 9 do not satisfy axiom (3) (see [11]).

(2) The geometry  $\circ \text{---}^3 \text{---} \circ \text{---}^2 \text{---} \circ \text{---}^c \text{---} \circ$  for  $\text{Aut HJ}$  is not self-dual since the residue of a block has 21 points whereas the residue of a block of the dual has 24 points.

There is another realization of the diagram  $\overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{3} \overset{c}{\circ}$ , namely a triple cover of the geometry given in Theorem 3 for  $\text{Aut } G_2(2)$ . This is due to the fact that  $G_2(2)$  has a simple subgroup  $\text{PSU}_3(9)$  of index 2.  $\text{PSU}_3(9)$  has a rank 4 representation on 36 points over  $\text{PSL}_3(2)$ , and an associated directed graph  $\Gamma$  on 36 vertices with in-valency 7 and out-valency 7 (e.g., take as vertices the points, lines, and flags of  $\text{PG}(2, 2)$ , and another vertex 0, and take as edges  $0 \rightarrow p$ ,  $p \rightarrow l$  [ $p \notin l$ ],  $p \rightarrow (p, l)$  [ $p \in l$ ],  $l \rightarrow 0$ ,  $l \rightarrow (p, l)$  [ $p = l \cap l'$ ],  $(p, l) \rightarrow p'$  [ $l = pp'$ ],  $(p, l) \rightarrow l$  [ $p \in l$ ],  $(p, l) \rightarrow (p', l')$  [ $p \in l \ni p' \in l', p \neq p', l \neq l'$ ]). Now take three copies  $X_1, X_2, X_3$  of the vertex set of  $\Gamma$ , and define the 3-partite graph  $\Gamma_3^*$  on  $X_1 \cup X_2 \cup X_3$  with edges  $x_i y_{i+1}$  iff  $x \rightarrow y, i \pmod 3$ . This graph  $\Gamma_3^*$  satisfies the conditions of Theorem 7. Hence it leads to a geometry with diagram  $\overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{3} \overset{c}{\circ}$ , and since it has a 3-partite point graph, also to a geometry with diagram



by Theorem 4.

Suzuki's graph for  $\text{Aut } G_2(2)$  is obtained by identifying all  $x_1, x_2, x_3$ . Notice the similarity with Proposition 5!

**Problems.** (1) Is there a geometry extending the sequence of geometries in Theorem 9 (resp. Theorem 13)?

(2) Classify all partial planes with lines containing 3 points such that if  $abc, ade, bef$  are lines, then  $cdf$  is a line.

(3) Are there geometries with diagram



related to HJ,  $G_2(4)$  and Sz?

(4) Are there any other applications of Theorem 7, with  $n \geq 3$ ?

(5) Are there geometries with diagram  $\overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{q} \overset{c}{\circ} \xrightarrow{q+1} \overset{c}{\circ}$ ? For  $q = 2, \overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{3} \overset{c}{\circ}$  is the same as  $\overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{3} \overset{c}{\circ}$ , and we have the above problem and examples. For  $q$  a prime power we have at least examples of  $\overset{c}{\circ} \xrightarrow{2} \overset{c}{\circ} \xrightarrow{q} \overset{c}{\circ} \xrightarrow{q+1} \overset{c}{\circ}$ , whose varieties are points and lines (rank 0), non-flags (rank 1), and flags (rank 2) of a projective plane of order  $q$ , with a suitable incidence defined on the varieties. The details are left to the reader.

We close with some remarks on Problem 4. If  $\Gamma_n$  satisfies the conditions of Theorem 7, then the neighbourhood of a  $(n - 2)$ -clique is a rectagraph with a

point- and edge-transitive group  $G$  of automorphisms such that  $G_x$  is fully 2-transitive on  $\Gamma(x)$ . It would be interesting to characterize such rectagraphs. Beside the  $n$ -dimensional cubes and the half-cubes (obtained by identifying opposite vertices of an  $n$ -dimensional cube with  $n \geq 5$ ), which realize  $H = A_n$  or  $S_n$ , I know of a number of sporadic examples: The Clebsch and Gewirtz graph (and their bipartite 2-cover, from Remark 3 of Section 1) realize  $H = A_5$  ( $k = 5$ ,  $v = 16$  or  $32$ ) and  $H = \text{PSL}_2(9)$  ( $k = 10$ ,  $v = 56$  or  $112$ ), and the incidence graphs of the biplanes  $B(4)$ ,  $B(5)$ , and  $B(6)$  of [4] realize  $H = S_4$  ( $k = 4$ ,  $v = 14$ ),  $H = A_5$  ( $k = 5$ ,  $v = 22$ ), and  $H = A_6$  or  $S_6$  ( $k = 6$ ,  $v = 32$ ). Except for the case  $H = S_4$ , discussed above, it is not known whether any of these extend to a  $\circ \text{---} \circ \text{---} \circ \text{---} \circ$ .

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## References

- [1] A.A. Bruen and J.W.P. Hirschfeld, Applications of line geometry over finite fields II, *Geom. Dedicata* 7 (1978) 333–353.
- [2] F. Buekenhout, Diagrams for geometries and groups, *J. Combin. Theory Ser. A* 27 (1979) 121–151.
- [3] F. Buekenhout and E. Shult, On the foundations of polar geometry, *Geom. Dedicata* 3 (1974) 155–170.
- [4] P.J. Cameron, Biplanes, *Math. Z.* 131 (1973) 85–101.
- [5] P.J. Cameron, Permutation groups with multiply transitive suborbits, *Proc. London Math. Soc.* III Ser. 25 (1972) 427–440.
- [6] P.J. Cameron, *Parallelisms of complete designs*, London Math. Soc. Lecture Note Ser. 23 (Cambridge Univ. Press, London, 1976).
- [7] X. Hubaut, Strongly regular graphs, *Discrete Math.* 13 (1975) 357–381.
- [8] M. Perkel, Bounding the valency of polygonal graphs with odd girth, *Canad. J. Math.* 31 (1979) 1307–1321.
- [9] M. Suzuki, A simple group of order 448, 345, 497, 600, in: *Theory of Finite Groups* (Benjamin, New York, 1969) pp. 113–119.
- [10] P. Wild, Semibiplanes, to appear.
- [11] F. Buekenhout, personal communication.