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ON A CLASS OF EDGE-REGULAR GRAPHS

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Summary

A regular graph is 4-regular if any two vertices at distance one resp. two have exactly λ resp. μ common neighbours. Some parameter relations are proved, and certain extremal cases of 4-regular graphs are classified. In particular, we obtain a characterization of double covers of complete graphs related to regular twographs.

All our graphs are finite, connected, undirected, without loops or multiple edges. We call the vertices of a graph points. The neighbourhood $\Gamma(x)$ of a point x of a graph Γ is the graph induced on the points adjacent with x; similarly $\Gamma_i(x)$ denotes the graph induced on the points at distance i from $x \in \Gamma$. A clique is a complete subgraph.

A graph Γ is edge-regular with parameters v, k, λ if there are v points, every point is adjacent with exactly k other points, and every edge is in exactly λ triangles. Γ is called 4-regular if it is edge-regular, and any two points at distance 2 have μ common neighbours (cf. Neumaier [8] for the general concept of t-regularity). The 4-regular graphs of diameter ≤ 2 are just the strongly regular graphs; they are well studied (see e.g. [5], [6], [10]). We call a 4-regular graph proper if its diameter is > 2.

Proposition 1
In a 4-regular graph, the number ℓ of points at distance two from a point x is independent of x and satisfies the relation

$$\ell\mu = k(k-1-\lambda) \qquad (1)$$

Proof. Count in two ways the number of paths of length 2. □

Example 1. Regular graphs of girth ≥ 5 ($\lambda=0$, $\mu=1$), and their line graphs ($k=2\lambda+2$, $\mu=1$). Note that for any 4-regular graph $\ell \leq k(k-1)$ with equality iff $\lambda = 0$, $\mu = 1$. More generally, point graphs and line graphs of 1-designs of girth ≥ 5; if there are K points on a line, and R lines through a point then the point graph has $\mu = 1$, $\lambda = K-2$, $k = R(K-1)$, and the line graph has $\mu = 1$, $\lambda = R-2$, $k = K(R-1)$. It is easy to see that every 4-regular graph with $\mu = 1$ arises in this way.

Example 2. The incidence graphs of semisymmetric designs (i.e. square designs where two points are in 0 or Λ blocks, and two blocks have 0 or Λ common points, see Wild [13]); they have $\lambda = 0$, $\mu = \Lambda$. The incidence graphs of symmetric 2-designs, symmetric nets, and partial Λ-geometries arise here. In particular, the symmetric complete bipartite graphs $K_{n,n}$ with a 1-factor deleted appear as the incidence graphs of a degenerate 2-$(n,n-1,n-2)$-design; here $\lambda = 0$, $\mu = n-2$. (Note that a free construction given by Cameron [3] shows that there are infinitely many infinite examples with $\lambda = 0$ of this type, for each k, μ).

Example 3. 4-regular graphs with $\lambda = 0$, $\mu = 2$ are called rectagraphs by Perkel [9], and are studied in Neumaier [7]. Examples are the n-cube, and the incidence graphs of biplanes. Cameron [4] gave a construction from linear binary codes of minimum weight ≥ 5: Vertices are the cosets, adjacent iff their union contains vectors at Hamming distance one.

Example 4. Every distance-regular graph (Biggs [1], [2]) is 4-regular.

Example 5. A double-cover of a graph Γ^* is a graph Γ such that there is a mapping $*$ of the vertices of Γ to the vertices of Γ^* such that every vertex of Γ^* has exactly two preimages, and $\Gamma(x)^* = \Gamma^*(x^*)$ for all $x \in \Gamma$. It is easy to see that all double covers of complete graphs arise in the following way: Take a graph Γ, a further copy Γ' of Γ, and two further vertices ∞ and ∞'. Add the edges ∞x for $x \in \Gamma$, xy' for $x,y \in \Gamma$, x nonadjacent with y, and $y'\infty'$ for $y \in \Gamma$. This defines a graph Γ^D. If Γ has v vertices then Γ^D is a double cover of K_{v+1}.

Proposition 2
A double cover Γ^D of a complete graph is 4-regular iff Γ is a strongly regular graph with parameters (v,k,λ,μ), where $k = 2\mu$. In this case Γ^D has parameters $k^D = \ell^D = v$, $\lambda^D = k$, $\mu^D = v-1-k$.

Proof. Let Γ^D be 4-regular. Suppose $x \in \Gamma$ has k neighbours in Γ. Then the edge ∞x of Γ^D is in k triangles whence $k = \lambda^D$, and Γ is regular. Let xy be a nonedge of Γ, and suppose that x and y have μ common neighbours in Γ. Then they have $\bar{\lambda} = v-2-2k+\mu$ common nonneighbours in Γ. Hence x and y have $1+\mu+\bar{\lambda} = v-1-2k-2\mu$ common neighbours in Γ^D. Therefore, $v-1-2k+2\mu = \mu^D$. But it is easy to see that $k^D = \ell^D = v$, whence by Prop. 1, $\mu^D = k^D-1-\lambda^D = v-1-k$; Therefore $2\mu = k$. In particular μ is constant.
Finally, let xy be an edge of Γ, and suppose that x and y have λ common neighbours in Γ_o. Then they have $\bar{\mu} = v-2k+\lambda$ common nonneighbours in Γ_o, i.e. xy is in $1+\lambda+\bar{\mu} = v-2k+2\lambda+1$ triangles of Γ^D. Hence $\lambda = \frac{1}{2}(\lambda^D-v+2k-1)$ is constant. So Γ is strongly regular, and satisfies $k = 2\mu$.
Conversely, if Γ is strongly regular and satisfies $k = 2\mu$ then the well-known relation $k(k-1-\lambda) = \mu(v-1-k)$ gives $\lambda = \frac{1}{2}(3k-1-v)$, and a reversal of the above arguments shows that Γ^D is 4-regular with the stated parameters. □

Note that an empty strongly regular graph (with $k=\lambda=\mu=0$) gives K_{vv} with a 1-factor deleted (cf. Example 2). The other 4-regular double covers of K_n are equivalent with regular twographs; see [11], [12]. In [12], Taylor and Levingston state that the double covers of Proposition 2 are in fact distance regular graphs, and he shows that they are characterized by their parameters. We prove the same characterization under the weaker assumption of 4-regularity:

Theorem 1

The parameters of a proper 4-regular graph Γ satisfy $\ell \geq k$. Equality holds iff Γ is a polygon, or a double cover of a complete graph.

Proof. It is clear that the proper 4-regular polygons and double covers of a complete graph satisfy $\ell = k$.
Conversely, let Γ be a proper 4-regular graph with $\ell \leq k$. We shall show that Γ is a polygon, or a double cover of a complete graph.

Step 1. $k = \ell = \lambda+\mu+1$. Moreover, if xy is an edge then every point of $\Gamma_2(x) \cap \Gamma_1(y)$ is adjacent with every point of $\Gamma_3(x) \cap \Gamma_2(y)$. –
For by (1), $\ell \leq k$ implies $k \leq \lambda+\mu+1$. To show equality, take $d \in \Gamma_3(x)$, $y \in \Gamma_1(x) \cap \Gamma_2(d)$. There are $k-1-\lambda$ points in $\Gamma_2(x) \cap \Gamma_1(y)$, but this set contains the $\mu \geq k-1-\lambda$ points of $\Gamma_1(y) \cap \Gamma_1(d)$. Hence the two sets coincide, and $\mu = k-1-\lambda$, i.e. by (1), $\ell = k$. If we let d vary over the points of $\Gamma_3(x) \cap \Gamma_2(y)$ we see that the second assertion is true.

Step 2. If $\mu = 1$ then Γ is a polygon. –
If $\mu = 1$ then $\lambda = k-2$ (Step 1!). Pick an edge xy, and denote by z the point adjacent to x but not to y (unique since $\lambda = k-2$). Every neighbour of $x \neq y,z$ is adjacent to both y and z, but y and z have only one common neighbour ($\mu=1$), which must be x. Hence y and z are the only neighbours of x. Therefore, $k=2$, $\lambda=0$, and Γ is a polygon.

Now let us fix two points a,d at distance 3. Define

$$B := \Gamma_1(a) \cap \Gamma_2(d), \quad A := \{x \in \Gamma_3(d) \mid \Gamma_1(x) \cap B \neq \emptyset\},$$

$$C := \Gamma_2(a) \cap \Gamma_1(d), \quad D := \{x \in \Gamma_3(a) \mid \Gamma_1(x) \cap C \neq \emptyset\}.$$

Of course $a \in A$, $d \in D$. The main effort consists in showing that $|A| = 1$, i.e. $A = \{a\}$, and we aim at this now.

Step 3. If $b \in B$ then $\Gamma_1(b) \subseteq A \cup B \cup C$; if $c \in C$ then $\Gamma_1(c) \subseteq B \cup C \cup D$. –
For if $b \in B$ then $b \in \Gamma_2(d)$; hence $z \in \Gamma_1(b)$ has distance 1,2, or 3 from d. If $z \in \Gamma_1(d)$ then $z \in \Gamma_2(a)$; hence $z \in C$. If $z \in \Gamma_2(d)$ then by Step 1 (for $x=a$, $y=b$), z cannot be nonadjcent with a; hence $z \in B$. If $z \in \Gamma_3(d)$ then $z \in A$ since $\Gamma_1(z) \cap B \supseteq \{b\} \neq \emptyset$. So $z \in A \cup B \cup C$. The second statement is symmetric to the first.

Step 4. $A - \{a\} \subseteq \Gamma_1(a)$. –
For if $a' \in A - \{a\}$ but $a' \notin \Gamma_1(a)$ then Step 1 (for $x=a$, $y \in \Gamma_1(a') \cap B$) gives a contradiction.

Step 5. Each point of $E := A \cup B - \{a\}$ has at most λ neighbours in E; an edge from $A - \{a\}$ to B is in $\lambda-1$ triangles of E. –
By Step 4, $E \subseteq \Gamma_1(a)$, and $\Gamma_1(a)$ has valency λ. This proves the first statement. If $a' \in A - \{a\}$ then $\Gamma_1(a') \cap C = \emptyset$, and if $b \in B \cap \Gamma_1(a')$ then $\Gamma_1(b) \subseteq A \cup B \cup C$. Hence the λ-set $\Gamma_1(a') \cap \Gamma_1(b)$ is in $A \cup B$; so $A \cup B - \{a\}$ contains $\lambda-1$ common neighbours of a' and b.

Step 6. If $a' \in A - \{a\}$ then $\Gamma_1(a) \cap \Gamma_1(a')$ is a λ-subclique of E. –
There is $b \in B \cap \Gamma_1(a')$. Define $A_o = (A-\{a\}) \cap (\Gamma_1(a') \cup \{a'\})$, $B_o = B \cap (\Gamma_1(b) \cup \{b\})$. They are in E. By Step 5, each neighbour of a' or b in E must be a neighbour of both. Hence $|A_o \cup B_o| = \lambda+1$, and a', b are adjacent with all other points of $A_o \cup B_o$. If we replace a' by $a'' \in A_o$ we find similarly that a'' is adjacent with all points of $A_o \cup B_o$, and if we replace b by $b' \in B_o$ we find that b' is adjacent with all points of $A_o \cup B_o$. Hence $A_o \cup B_o$ is a clique. But $\Gamma_1(a) \cap \Gamma_1(a') \supseteq A_o \cup B_o \setminus \{a'\}$, and since both sets have size λ, they are the same.

Step 7. If $|A| > 1$ then $\mu \leq \lambda$. –
Take $a' \in A-\{a\}$, $b \in \Gamma_1(a') \cap B$, $c \in \Gamma_1(b) \cap C$. Then a' is adjacent with a and has distance 2 from c. Neighbours of c are in $B \cup C \cup D$, whence $\Gamma_1(a') \cap \Gamma_1(c) \subseteq B \subseteq \Gamma_1(a)$. Hence $\Gamma_1(a') \cap \Gamma_1(c) \subseteq \Gamma_1(a') \cap \Gamma_1(a)$, which implies $\mu \leq \lambda$.

Step 8. If $|A| > 1$ then $\Gamma_1(a)$ is a $(\lambda+1)$-clique. –
$\Gamma_1(a)$ contains the $(\lambda+1)$-clique $(\Gamma_1(a) \cap \Gamma_1(a')) \cup \{a'\}$ constructed in Step 6. Since the valency of $\Gamma_1(a)$ is λ, the $(\lambda+1)$-clique is a component.

If $\Gamma_1(a)$ were not a $(\lambda+1)$-clique, there were another component of valency λ, hence size $\geq \lambda+1 > \mu$ (by Step 7). The size of $\Gamma_1(a)$ would be bigger then $(\lambda+1)+\mu = k$, contradiction.

Step 9. $|A| = 1$. -
Assume that $|A| > 1$. Since a polygon has $|A| = 1$, Γ is not a polygon, hence $\mu > 1$. Choose $c \in C$, and let b, b' be two of the μ points adjacent with a and c. Then $b, b' \in \Gamma_1(a)$, hence by Step 8, they are adjacent and have $\lambda-1$ common neighbours in $\Gamma_1(a)$. But they also have the neighbours $a, c \notin \Gamma_1(a)$, contradiction.

Step 10. If $B \neq \Gamma_1(a)$ then Γ is a polygon. -
Choose $e \in \Gamma_1(a) \setminus B$. Then $\Gamma_1(e) \cap B = \emptyset$ (Step 3, first part) and $\Gamma_1(e) \cap C = \emptyset$ (Step 3, second part). Moreover, for any $b \in B$, $\Gamma_1(b) \subseteq A \cup B \cup C$, whence $\Gamma_1(b) \cap \Gamma_1(e) \subseteq A$. But since $\Gamma_1(e) \cap B = \emptyset$, the left hand side has size $\mu > 0$. Hence by Step 9, $\mu = 1$, and by Step 2, Γ is a polygon.

Step 11. If $B = \Gamma_1(a)$ then Γ is the double cover of a complete graph. -
In this case, by definition of B, $\Gamma_1(a) \subseteq \Gamma_2(d)$. Hence by Step 3, $\Gamma_2(a) \subseteq C \subseteq \Gamma_1(d)$. Since $|\Gamma_2(a)| = \ell = k = |\Gamma_1(d)|$ we have $\Gamma_2(a) = \Gamma_1(d)$. Now a point of $C = \Gamma_2(a) = \Gamma_1(d)$ has μ neighbours in B, λ neighbours in C, and the neighbour d; since $k = \mu+\lambda+1$, there are no further neighbours. Also d has all its neighbours in $\Gamma_1(d) = C$. Hence $\{a\} \cup \Gamma_1(a) \cup \Gamma_2(a) \cup \{d\}$ is a component of Γ, and since Γ is connected, this is Γ. Therefore, Γ contains exactly $1+k+k+1$ points. Hence each point x has a unique point $\bar{x}$ at distance 3 with x. Hence Γ is a double cover of a complete graph K_{k+1} (the mapping $*$ identifies each x with $\bar{x}$). □

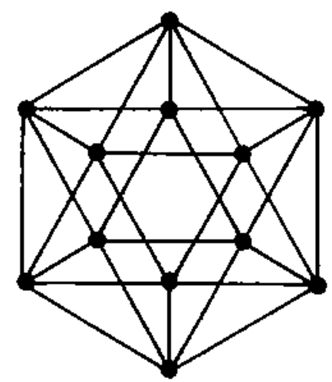
A second inequality of Taylor and Levingston [12] for distance regular graphs also generalizes to 4-regular graphs. Again the case of equality can be characterized.

Theorem 2
A proper 4-regular graph Γ satisfies $k \geq 2\lambda+3-\mu$, with equality iff Γ is either the icosahedron ($k=5$, $\lambda=\mu=2$), or the line graph of a regular graph of girth ≥ 5 ($k=2\lambda+2$, $\mu=1$).

Proof. Let xty be an induced path of length 2. Denote by q the number of induced quadrangles containing xty, and by c the number of 3-claws

 containing xty. The $k-2$ vertices $z \neq x,y$ adjacent with t fall into four classes: There are $\mu-1-q$ such vertices adjacent with x and y, $k-2-\lambda-c$ such vertices adjacent with x but not with y, $k-2-\lambda-c$ such vertices z adjacent with y but not with x, and c vertices z adjacent with neither of x,y. Hence $k-2 = (\mu-1-q) + 2(k-2-\lambda-c) + c$, or $k = 2\lambda+3-\mu+q+c$. Hence $k \geq 2\lambda+3-\mu$, and equality implies that Γ contains neither 3-claws nor induced quadrangles.

Now assume that $k = 2\lambda+3-\mu$. If $\mu=1$ then by the remark of Example 1, Γ is the line graph of a regular graph of girth ≥ 5. If $\mu > 1$ then $k \leq 2\lambda+1$. Choose $a \in \Gamma$, and consider $\Gamma(a)$. $\Gamma(a)$ is regular of valency λ, not a clique, and has $k \leq 2\lambda+1$ points. Also, $\Gamma(a)$ contains neither 3-cocliques nor induced quadrangles. Hence the complement of $\Gamma(a)$ is triangle-free, and has no induced subgraph of the shape  ; in particular it contains no induced n-gon with $n \geq 6$. If it contains a pentagon then it is a pentagon (any extra vertex produces a ); otherwise it is easy to see that it is a complete bipartite graph. Hence $\Gamma(a)$ itself is either a pentagon or a union of two cliques. But the latter is impossible since $k \leq 2\lambda+1$. Hence each $\Gamma(a)$ is a pentagon, $k=5$, $\lambda=\mu=2$, and it is a well-known exercise that then Γ is the icosahedron. □



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