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Distance matrices, dimension, and conference graphs

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SUMMARY

We define the dimension of a distance matrix and its associated metric space, and use this to give necessary and sufficient conditions for a metric space to be isometrically embeddable into suitable real inner product spaces and Euclidean spheres. Also, for certain distance matrices C with irrational entries, we derive the bound $w \leq 2f+1$ for the size w of C in terms of its dimension f. This result is applied to improve a bound by Larman, Rogers, and Seidel on two-distance sets in Euclidean space, and to characterize certain regular graphs as conference graphs.

Let X be a finite subset of a metric space with distance function $d(x,y)$. Then the matrix $C=(d(x,y)^2)_{x,y\in X}$ is called the *distance matrix* of X (this differs slightly from the definition in Menger [2]). An (abstract) *distance matrix* (see Neumaier [3]) is a nonzero, real, symmetric matrix $C=(c_{xy})$ with nonnegative entries and zero diagonal such that the function $d(x,y)=\sqrt{c_{xy}}$ satisfies the triangle inequality $d(x,y)+d(y,z)\geq d(x,z)$. The relation $x\equiv y$ if $d(x,y)=0$, defined on the set X of rows of C, is an equivalence relation, and equivalent rows have identical entries. Hence C has no off-diagonal zero entries iff C has no repeated rows iff the rows of C form a metric space.

We say that a distance matrix $C=(c_{xy})$ (or an associated metric space X) is *isometrically embeddable* into a metric space S if there is a map $x\to p_x$ of the set X of rows of C into S such that for all $x,y\in X$, $c_{xy}=d(p_x,p_y)^2$. Then $x\to p_x$ is called an *embedding* of C (or X) into S. An embedding of a distance matrix C is injective iff C has no repeated rows.

The *dimension* of a distance matrix C with w rows (or an associated finite metric space) is the rank f of the matrix $G=-(I-w^{-1}J)C(I-w^{-1}J)$; here I denotes the identity matrix, and J is the all-one matrix of size w. This definition is motivated by the following

THEOREM 1

Let C be a distance matrix of dimension f. Then there is an f-dimensional real inner product space V, and there are points $p_x \in V (x \in X)$ such that

(1) $\quad c_{xy} = |p_x - p_y|^2$ for all $x, y \in X$;

here X is the set of rows of C, and $|p|^2 = (p,p)$.

PROOF. Suppose that C has w rows. Let $\{e_x | x \in X\}$ be the standard basis of $\mathbb{R}^w$, and let V be the subspace of $\mathbb{R}^w$ consisting of the row vectors aG, where $a \in \mathbb{R}^w$ (we assume that $\mathbb{R}^w$ consists of row vectors). Then V has dimension f and is generated by the rows of $G, p_x = e_x G, (x \in X)$. Now the expression $\frac{1}{2}aGb$ depends only on aG and bG, and hence defines an inner product $(.,.)$ on V such that for $x, y \in X, 2(p_x, p_y)$ is the (x,y)-entry of G. So, by definition of G,

(2) $\quad 2(p_x, p_y) = -s + s_x + s_y - c_{xy}$ for all $x, y \in X$,

where

(3) $\quad s = w^{-2} \sum_{x,y} c_{xy}, \qquad s_x = w^{-1} \sum_{y} c_{xy}.$

Now $GJ = 0$, whence

(4) $\quad \sum_x p_x = 0.$

Therefore the well-known identity

(5) $\quad |p_x - p_y|^2 = |p_x|^2 + |p_y|^2 - 2(p_x, p_y),$

together with (3), implies

(6) $\quad s = 2w^{-1} \sum_{y} |p_y|^2, \qquad s_x = |p_x|^2 + w^{-1} \sum_{y} |p_y|^2,$

so that (2) implies (1).

THEOREM 2

Any two embeddings of a distance matrix C into a real inner product space are congruent; in particular, they span spaces of the same dimension $f = \dim(C)$.

PROOF. Let $x \to p_x$ be an embedding of C into $\mathbb{R}^n$ with inner product $(\ ,\ )$, so that (1) holds. Since translations are congruences, we may assume that the centre of mass of the p_x is in the origin, so that (4) holds. Then also (6) holds, which, together with (5) and (1), imply $2(p_x, p_y) = |p_x|^2 + |p_y|^2 - |p_x - p_y|^2 = (s_x - \frac{1}{2}s) + (s_y - \frac{1}{2}s) - c_{xy}$, so that (2) holds. Hence $2(p_x, p_y)$ is the (x,y)-entry of G, whence we obtain a congruence of the space spanned by the p_x and the row space of G which maps the p_x onto the points labelled p_x in the proof of Theorem 1.

COROLLARY 1 (Seidel's condition [5])

A distance matrix C is isometrically embeddable into a Euclidean space $\mathbb{R}^n$ iff dim $(C)\leq n$ and G is positive semidefinite. For $n=\dim(C)$, this embedding is unique up to congruences.

We call a distance matrix C *Euclidean* if $G=-(I-w^{-1}J)C(I-w^{-1}J)$ is positive semidefinite.

REMARKS

1. Define the matrices $G_a=(c_{ax}+c_{ay}-c_{xy})_{x,y\in X-\{a\}}$ and $G_0=\begin{pmatrix}0 & j^T\\ j & C\end{pmatrix}$, where j denotes an all-one vector of size w. G_0 is what Menger [2] calls a distance matrix. If C has dimension f then G_a has rank f and G_0 has rank $f+2$, and C is Euclidean iff the following two equivalent conditions are satisfied:

(i) G_a is positive semidefinite for some (hence all) $a\in X$ (Schoenberg's condition [4]),

(ii) G_0 has exactly one positive eigenvalue, and this eigenvalue is simple (Menger's condition [2]).

2. Note that $GJ=0$, whence $f\leq w-1$.

Most of the distance matrices arising in applications are isometrically embeddable into a Euclidean sphere $S^n(r)=\{x\in\mathbb{R}^n \mid |x|=r\}$, with the distance induced by the Euclidean distance of $\mathbb{R}^n$. Such distance matrices (and associated finite metric spaces) are called *spherical.*

THEOREM 3

Let C be a distance matrix of dimension f. C is spherical iff, for some $\gamma>0$, the matrix $H=2\gamma J-C$ is positive semidefinite. If γ is chosen minimally then C is isometrically embeddable into a Euclidean sphere $S^f(r)$ with $r=\sqrt{\gamma}$.

PROOF. Let $x\to p_x$ be an embedding of the rows of C into $S^n(r)$. Then $c_{xy}=|p_x-p_y|^2$ and $|p_x|=r$, whence by (5), $2(p_x,p_y)=2r^2-c_{xy}$. Hence $2r^2J-C=2G$, where $G=((p_x,p_y))$ is the positive semidefinite Gram matrix of the p_x. Conversely, let H be positive semidefinite of rank n. Then $G=\frac{1}{2}H$ is the Gram matrix of certain points p_x of $\mathbb{R}^n$, and $2\gamma-c_{xy}=2(p_x,p_y)$. Hence $|p_x|^2=(p_x,p_x)=\gamma$, whence the p_x are points of $S^n(r)$ with $r=\sqrt{\gamma}$. Moreover, by (5), $|p_x-p_y|^2=|p_x|^2+|p_y|^2-2(p_x,p_y)=c_{xy}$ so that $x\to p_x$ is an embedding into $S^n(r)$. Now let γ be minimal. The map $x\to p_x$ is an embedding into $\mathbb{R}^n$. By Theorem 2, the p_x span a f-dimensional space, and hence are contained in a f-dimensional section $S^f(r')$ of $S^n(r)$. By minimality of $\gamma, r'=r$.

COROLLARY 2

If H is a positive semidefinite symmetric matrix with constant diagonal mI, and $H\neq mJ$, then $C=mJ-H$ is a spherical distance matrix.

We continue with some consequences of the above theorems. If the row sums of a distance matrix C all have the same value c, $CJ=cJ$, we say that C has *strength 1* (see Neumaier [3] who also defines distance matrices of strength t for $t>1$). In this case the formula G simplifies to $G=w^{-1}cJ-C$.

PROPOSITION 1

Let X be a finite set of points of a Euclidean space. Then the distance matrix C of X has strength 1 iff X is contained in a sphere around the centre of mass of X.

PROOF. By a suitable translation we may assume that the centre of mass of X is in the origin. Then (3) and (6) are valid, and they imply that C has strength 1 iff C has constant row sums iff $s_x=\text{const.}$ iff $|p_x|=$ const. iff the points of X are on a sphere around the origin.

COROLLARY 3

A Euclidean distance matrix of strength 1 is spherical.

COROLLARY 4 (Seidel [6])

Let X be a finite spanning set of points on a sphere around 0. Then the distance matrix of X has strength 1 iff the centre of mass of X is in 0.

PROPOSITION 2

Let C be a distance matrix with smallest eigenvalue $-n$. Then $C'=n(J-I)-C$ is a spherical distance matrix. C' has strength 1 iff C has strength 1.

PROOF. $H=nJ-C'=C+nI$ is positive semidefinite, with constant diagonal nI. Hence by Corollary 2, C' is a spherical distance matrix. The remark on the strength is obvious.

REMARK. It follows from [3], Theorem 2.2 (ii) that C' has strength 2 iff C has strength 2; but by [3], 2.5, the corresponding result for strength $t>2$ is not true in general.

The remainder of the paper deals with combinatorial aspects of distance matrices.

THEOREM 4

Let C be a distance matrix of dimension f with $w\geq\max(2f+1,6)$ rows. Suppose that $C=m(J-I)-M$ with an integral matrix M. Then either m is integral, or $w=2f+1$. In the latter case there are integers p and q such that the matrix $N=(I-w^{-1}J)M(I-w^{-1}J)$ satisfies

(7) $$N^2=pN+q(I-w^{-1}J),\quad tr(N)=pf.$$

PROOF. f is the rank of $G=-(I-w^{-1}J)C(I-w^{-1}J)=N+m(I-w^{-1}J)$. Hence N has an eigenvalue 0 belonging to the eigenvector j, the eigenvalue $-m$

with multiplicity $w-1-f$, and f other eigenvalues (maybe repeated). We show that $-m$ is also an eigenvalue of M. In fact, $J=jj^T$, whence $Nx-Mx=$ $=-w^{-1}j(j^TMx)-w^{-1}Mj(j^Tx)+w^{-2}j(j^TMj)(j^Tx)$ is a linear combination of j and Mj. Hence in any 3-dimensional subspace of $\mathbb{R}^w$ there is a nonzero vector x with $Nx=Mx$. Since $w\geq \max(2f+1,6)$, $w-1-f\geq 3$, the eigenspace of the eigenvalue $-m$ of N contains a nonzero vector x with $-mx=Nx=Mx$, and $-m$ is an eigenvalue of M. Now M is an integral matrix, whence $-m$ is an algebraic integer, hence either integral or irrational. If $-m$ is irrational then it has a conjugate $-\bar{m}\neq -m$, which must be an eigenvalue of N with multiplicity $w-1-f\leq f$, since only f eigenvalues are left. Now our assumptions imply $w=2f+1$, and the only eigenvalues of N are $-m$, $-\bar{m}$, and the simple eigenvalue 0 belonging to j. Hence the minimal polynomial of m is quadratic, say x^2-px-q, with integers p and q since m is an algebraic integer, and $p=-m-\bar{m}$, $q=-m\bar{m}$. Moreover, $(N+mI)(N+\bar{m}I)=w^{-1}m\bar{m}J$. The trace of N is the sum of all eigenvalues of N weighed with their multiplicities, hence $tr(N)=-mf-\bar{m}f$. This proves (7).

REMARK. (7) implies that $G=N+m(I-w^{-1}J)$ is a eutactic star (see Seidel [6] for a definition). Examples of the described situation with irrational m are the distance matrices of conference graphs; see the proof of Theorem 6.

An (abstract) *s-distance set* is a metric space X with the property that there are exactly s possible distances between distinct points of X.

COROLLARY 5

Let X be an f-dimensional two-distance set, with distances α, $\beta (\alpha<\beta)$. If X contains more than max $(2f+1, 5)$ points then $\alpha^2/\beta^2=(m-1)/m$ with an integer $m\geq 2$.

PROOF. Write $m=\beta^2/(\beta^2-\alpha^2)$ so that $\alpha^2/\beta^2=(m-1)/m$. Then the distance matrix of X is $(\beta^2-\alpha^2)(m(J-I)-M)$, where the matrix M has (x,y)-entry 1 if $d(x,y)=\alpha$, and 0 otherwise. Hence also $C=m(J-I)-M$ is a distance matrix, and M is integral. By our assumptions and Theorem 4, m is an integer.

REMARKS

1. This improves a result by Larman, Rogers, and Seidel [1], who, in case that X is a subset of a Euclidean space, obtained the result only under the assumption $|X|>2f+3$.

2. Because of Theorem 1, the proof of Theorem 1 of [1] remains valid for arbitrary metric spaces. Hence any abstract two-distance set of a dimension f contains at most $\frac{1}{2}(f+1)(f+4)$ points. Can this bound be attained?

For the last result we recall some definitions (see e.g. Seidel [7]). A graph Γ (undirected, without loops or multiple edges) with vertex set X of size n is called *regular* if every vertex is adjacent to exactly k other vertices, and *strongly regular* if, in addition, the number of vertices adjacent to two distinct vertices x and y is λ or μ depending on whether x and y are adjacent or not. The *adjacency* *matrix* of a graph on X is the matrix $M=(m_{xy})_{x,y\in X}$ with $m_{xy}=1$ if x and y are adjacent, and $m_{xy}=0$ otherwise. Regularity implies

(8) $\quad MJ=kJ,$

and for a regular graph, strong regularity is equivalent to

(9) $\quad M^2=(\lambda-\mu)M+(k-\mu)I+\mu J.$

A *conference graph* is a strongly regular graph with parameters
$n=4\mu+1, k=2\mu, \lambda=\mu-1.$

THEOREM 5

Let $-m$ be the smallest eigenvalue of the adjacency matrix M of a regular, connected graph Γ. Unless Γ is the complete graph, $C=m(J-I)-M$ is a spherical distance matrix. The dimension of C is $f=n-1-g$, where g is the multiplicity of $-m$ as an eigenvalue of M.

PROOF. By Corollary 2, applied to $H=mJ-C=M+mI$, C is a spherical distance matrix, unless Γ is complete. Now

$$G=n^{-1}cJ-C=(n^{-1}c-m)J+M+mI.$$

Since j is an eigenvector for the largest, simple eigenvalue k, all eigenvectors x of M for an eigenvalue $\theta\neq k$ are orthogonal to j, hence satisfy $Jx=0$. Therefore $Gx=(\theta+m)x$. Since $\theta+m\geq 0$, the kernel of G consists of the vector space spanned by j and the eigenspace for the eigenvalue $-m$ of M. Hence the rank of G is $n-1-g$.

THEOREM 6

Let Γ be a regular graph with $n\geq 6$ vertices, whose adjacency matrix has smallest eigenvalue $-m$ with multiplicity g.

(i) If Γ is strongly regular then m is integral, or $n=2g+1$ and Γ is a conference graph.
(ii) If $n<2g+1$ then m is integral.
(iii) If $n=2g+1$ then m is integral, or Γ is a conference graph.

PROOF. (i) is well-known – see e.g. Seidel [7]. The distance matrix $C=m(J-I)-M$ of Γ has dimension $f=n-1-g$. If $n<2g+1$ then $n>2f+1$ and, by Theorem 4, m is integral. If $n=2g+1$ then $n=2f+1$, and if m is not integral then, again by Theorem 4, the matrix $N=M-n^{-1}kJ$ (simplified with (8)) satisfies (7). Therefore, (9) holds for certain parameters λ, μ, and Γ is strongly regular. Hence by (i), Γ is a conference graph.

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