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Classification of Graphs by Regularity

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We give a classification of graphs by two parameters s and t such that a graph is regular iff $t \geqslant 2$, edge-regular iff $t \geqslant 3$, and distance regular of diameter δ iff $s = \delta$, $t \geqslant 2\delta - 2$. We investigate the algebra of polynomials in the adjacency matrix and relate to every graph a family of orthogonal polynomials. This generalizes various results on distance regular graphs.

0. INTRODUCTION

Among the regular graphs, certain classes have received much attention in the past: Strongly regular graphs (see, e.g., Bose [2], Hubaut [5], Seidel [6]), edge-regular graphs (e.g., Bose and Laskar [3]), and distance regular graphs (see Biggs [1], Delsarte [4]). The object of this paper is to show that all graphs can be classified in such a way that the above classes are extremal in our classification. Also, certain properties of distance regular graphs can be generalized to arbitrary graphs.

In view of applications in subsequent papers we state our theory in terms of Fiedler matrices. A Fiedler matrix is a nonzero symmetric matrix with zero row sums and off-diagonal elements $\leqslant 0$. Fiedler matrices are always positive semidefinite. The number s of distinct positive eigenvalues of B is called the geometric rank of B. Another invariant t is defined by properties of the pointwise product of polynomials in B and measures the inner regularity of B.

Various calculations give insight into the algebraic structure of the B-algebra, i.e., the algebra of polynomials in B. This algebra can be described in terms of a special basis $D_0 = I, D_1, ..., D_s$, and a related family $p_i(x)$ of orthogonal polynomials. If $t \geqslant 2e$, then $D_1, ..., D_e$ are (0, 1)-matrices with zero diagonal, and if $t \geqslant 2s - 2$, then D_1 is the adjacency matrix of a distance regular graph Γ of diameter s. In this case, the B-algebra is the adjacency algebra of Γ.

If Γ is a graph with vertex set X, then we define a matrix $B = (b_{xy})_{x,y \in X}$ with $b_{xy} =$ valency of x if $x = y$, $b_{xy} = -1$ if x and y are adjacent, and

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$b_{xy} = 0$ otherwise. B is a Fiedler matrix, and Γ is called t-regular of rank $s+1$ if B has parameters s and t. s is not less than the diameter of Γ, with equality, e.g., in the distance regular case. t-regularity is equivalent with the constancy of certain parameters p_{ij} of the graph. In particular, a graph Γ is regular (edge-regular, strongly regular) iff it is 2-regular (3-regular, 3-regular of rank 3), and distance regular of diameter s iff it is $(2s-2)$-regular.

Remark. The present classification is formally dual to the classification of distance matrices (Neumaier [9]).

Notation. We denote the identity matrix of any size by I, an all-one vector of any size by j, and an all-one matrix of any size by J. If $A = (a_{xy})$ and $B = (b_{xy})$ are $v \times v$-matrices, then we denote by $A \circ B = (a_{xy} b_{xy})$ the pointwise product of A and B. In particular, $A \circ J = A$ for all matrices A. We also make use of the Kronecker symbol $\delta_{ik} = 1$ if $i = k$, $\delta_{ik} = 0$ otherwise. For a $v \times v$-matrix B, the B-algebra is defined as the algebra of polynomials in B.

1. Fiedler Matrices

Let X be a v-set. A *Fiedler matrix* on X is a symmetric $v \times v$-matrix $B = (b_{xy})_{x,y \in X}$ with zero row sums and off-diagonal entries $\leqslant 0$ (cf. Fiedler [8]). A Fiedler matrix on X is called *connected* if it is impossible to split X into two nonempty disjoint subsets Y, Z such that $b_{yz} = 0$ for $y \in Y$, $z \in Z$.

1.1. Theorem. *Every Fiedler matrix B is positive semidefinite, and the all-one vector j is eigenvector to the eigenvalue $\alpha = 0$. B is connected iff* 0 *is a simple eigenvalue of B.*

Proof. Obviously, $Bj = 0$. Let u be an eigenvector for the eigenvalue α of B. Then u has some nonzero entry, and we can normalize u such that the maximal positive entry is 1. Define $Y = \{y \in X \mid u_y = 1\}$, $Z = X \backslash Y$. Then $Y \neq \varnothing$, and for $y \in Y$, $\alpha = \alpha u_y = \sum_{z \in X} b_{yz} u_z \geqslant \sum_{z \in X} b_{yz} = 0$ since $b_{yz} \leqslant 0$, $u_z \leqslant 1$ for $z \neq y$, and $u_y = 1$. Hence $\alpha \geqslant 0$, and B is positive semidefinite. Moreover, $\alpha = 0$ implies that $b_{yz} u_y = b_{yz}$ for all $y \in Y$, $z \in X$, whence $b_{yz} = 0$ for all $y \in Y$, $z \in Z$. If B is connected, then $Z = \varnothing$, $Y = X$, whence $u = j$, so 0 is a simple eigenvalue. If B is not connected, we may split X into nonempty disjoint subsets Y, Z with $b_{yz} = 0$ for $y \in Y$, $z \in Z$. Then the vector u with $u_x = 1$ if $x \in Y$, $u_x = 0$ if $x \in Z$ satisfies $Bu = 0$, and is independent from j, whence 0 is not a simple eigenvalue of B.

We write S for the set of nonzero eigenvalues of B, and $S' = S \cup \{0\}$. The multiplicity of the eigenvalue $\alpha \in S'$ is denoted by f_α, so that $f_0 = 1$ iff B is connected. The number $s = |S|$ of distinct nonzero eigenvalues of B is called the *geometric rank* of B. We also introduce the *annihilator polynomial* $\mathrm{Ann}_B(x)$ of B by

$$\mathrm{Ann}_B(x) = v \sum_{\alpha \in S} \left(1 - \frac{x}{\alpha}\right). \tag{1.1}$$

1.2. Lemma. *The minimal polynomial of a Fiedler matrix B is a scalar multiple of $x\,\mathrm{Ann}_B(x)$. Moreover, B is connected iff*

$$\mathrm{Ann}_B(B) = J. \tag{1.2}$$

Proof. The minimal polynomial of a symmetric matrix B has as zeros just the eigenvalues of B, all simple. Hence the first result follows from the definition of $\mathrm{Ann}_B(x)$. If B is not connected, and Y, Z are as above, then the entry of B^i at a place $(y, z) \in Y \times Z$ is zero, for all i (induction). In particular, $\mathrm{Ann}_B(B)$ contains zero entries and (1.2) does not hold. If B is connected, then $H = \mathrm{Ann}_B(B)$ is symmetric and satisfies $BH = 0$. Hence the rows of H are multiples of j whence H is a multiple of J. Now $HJ = vJ = J^2$ implies $H = J$.

For a Fiedler matrix B, we denote the B-algebra by V. By Lemma 1.2, V had dimension $s + 1$. Let us define the polynomials

$$A_\alpha(x) = v \prod_{\gamma \in S' \setminus \{\alpha\}} \left(\frac{\gamma - x}{\gamma - \alpha}\right), \qquad \alpha \in S'. \tag{1.3}$$

Then for $\alpha, \beta \in S'$,

$$A_0(x) = \mathrm{Ann}_B(x), \tag{1.4a}$$

$$A_\alpha(x) = \frac{vx\,\mathrm{Ann}_B(x)}{\alpha\,\mathrm{Ann}_B{}'(\alpha)(x - \alpha)} \qquad \text{if} \quad \alpha \neq 0, \tag{1.4b}$$

$$A_\alpha(\beta) = v\delta_{\alpha\beta}, \tag{1.5a}$$

$$A_\alpha(x)\,A_\beta(x) \equiv v\delta_{\alpha\beta}A_\alpha(x) \qquad \mathrm{mod}\ \mathrm{Ann}_B(x), \tag{1.5b}$$

and we have

1.3. Theorem. *The matrices $J_\alpha = A_\alpha(B)$, $\alpha \in S'$ form a basis of the B-algebra V, and satisfy for $\alpha, \beta \in S'$,*

$$J_\alpha J_\beta = vd_{\alpha\beta}J_\alpha, \tag{1.6}$$

$$\mathrm{tr}(J_\alpha) = vf_\alpha, \tag{1.7}$$

$$p(B) = v^{-1} \sum_{\alpha \in S'} p(\alpha)\,J_\alpha \qquad \text{for polynomials } p(x), \tag{1.8}$$

$$\sum_{\alpha \in S'} J_\alpha = vI. \tag{1.9}$$

Moreover, if B is connected, then

$$J_0 = J, \tag{1.10}$$

$$p(B)J = p(0)J. \tag{1.11}$$

Proof. Equation (1.5b) implies (1.6). From (1.3), J_α has the eigenvalue $A_\alpha(\alpha) = v$ with multiplicity f_α, and other eigenvalues zero. The trace is the sum of all eigenvalues weighed with their multiplicities, whence (1.7) holds. $A = p(B) - v^{-1} \sum_{\alpha \in S'} p(\alpha) J_\alpha$ is a symmetric matrix whose eigenvalues are all zero, whence $A = 0$, and (1.8) follows. Equation (1.9) is the special case $f(x) = v$ of (1.8). Equation (1.10) follows from (1.4a) and Lemma 1.2, and (1.11) from (1.10), (1.8), and (1.6).

2. Regularity

Let B be a fixed connected Fiedler matrix of geometric rank s, and V be the corresponding B-algebra. For $i = 0,..., s$, V_i denotes the subspace of V consisting of all polynomials in B of degree at most i. Obviously, V_i has dimension $i + 1$, and $V_0 \subset V_1 \cdots V_s = V$.

2.1. Lemma. *There are unique matrices $E_0,..., E_s$ satisfying*

$$V_i = \langle E_0,..., E_i \rangle \qquad \text{for} \quad i = 0,..., s, \tag{2.1}$$

$$j^T(E_i \circ E_k)j = \delta_{ik} \qquad \text{for} \quad i, k = 0,..., s. \tag{2.2}$$

Proof. Define on V an inner product $(A, B) = j^T(A \circ B)j$. This is the canonical inner product on $v \times v$-matrices considered as v^2-dimensional vectors, whence it is positive definite. Hence, by the Gram–Schmidt algorithm there is a unique basis $E_0,..., E_s$ of V, orthonormal with respect to (,) and satisfying (2.1).

2.2. Theorem. *There are nonnegative numbers d_i and polynomials $p_i(x)$ of degree i ($i = 0,..., s$) such that the matrices*

$$D_i = p_i(B) = v^{-1} \sum_{\alpha \in S'} p_i(\alpha) J_\alpha \qquad (i = 0,..., s) \tag{2.3}$$

satsisfy for $i, k = 0,..., s$ the relations

$$j^T(D_i \circ D_k)j = \delta_{ik}(j^T D_i j), \tag{2.4}$$

$$D_i J = d_i J, \qquad p_i(0) = d_i, \tag{2.5}$$

$$D_0 = I, \qquad p_0(x) = 1, \qquad d_0 = 1. \tag{2.6}$$

Moreover, if $d_i = 0$, then $D_i = 0$ and $p_i(x) = 0$. (Here the zero polynomial is assumed to have arbitrary degree.)

Proof. Define $e_i = j^T E_i j$, and $D_i = e_i E_i$. Then $j^T(D_i \circ D_k) j = e_i e_k j^T(E_i \circ E_k) j = \delta_{ik} e_i^2 = \delta_{ik} j^T D_i j$, i.e., (2.4) holds. Since E_i is a polynomial of degree i in B, so is D_i, and (2.3) follows from (1.8). Equation (2.5) follows from (1.11), and (2.6) directly from $D_0 = p_0(B) = d_0 I$ and (2.4). Finally, (2.5) implies $d_i v = j^T D_i j = e_i j^T E_i j = e_i^2 \geqslant 0$, whence $d_i \geqslant 0$, with equality iff $e_i = 0$, i.e., $D_i = 0$.

2.3. LEMMA

(i) $\sum_{i=0}^{s} D_i = J$, $\quad \sum_{i=0}^{s} p_i(x) = \mathrm{Ann}_B(x)$, $\quad \sum_{i=0}^{s} d_i = v$.

(ii) $d_s > 0$, $\quad D_s \neq 0$.

(iii) *V is generated by $J, E_0, \ldots, E_{s-1}$.*

Proof. $J = \mathrm{Ann}_B(B) \in V_s \backslash V_{s-1}$ since $\mathrm{Ann}_B(x)$ has degree s. Hence in the representation $J = \sum_{i=0}^{s} a_i E_i$ we have $a_s \neq 0$. Now with e_i as before, $e_i = j^T E_i j = j^T(E_i \circ J) j = a_i$ by (2.2). Therefore $J = \sum_{i=0}^{s} e_i E_i = \sum_{i=0}^{s} D_i$, and $e_s \neq 0$. Now (i) follows from (2.3) and (2.5), (ii) since $d_i v = e_i^2$, and (iii) from (i) since $E_s = e_s^{-1}(J - \sum_{i=0}^{s-1} e_i E_i)$.

2.4. THEOREM. *The following conditions are equivalent for any t:*

(i) *$B^i \circ B^k = f_{ik}(B)$ with a polynomial $f_{ik}(x)$ of degree $\leqslant \min(i, k)$, for $i + k \leqslant t$, $i, k \leqslant s$.*

(ii) *$V_i \circ V_k \subseteq V_{\min(i,k)}$ for $i + k \leqslant t$, $i, k \leqslant s$.*

(iii) *$E_i \circ E_k = (\mathrm{const})\, \delta_{ik} E_k$, for $i + k \leqslant t$, $i, k \leqslant s$, $\mathrm{const} \neq 0$.*

They imply

(iv) *$D_i \circ D_k = \delta_{ik} D_k$ for $i + k \leqslant t$, $i, k \leqslant s$, and $d_i > 0$ for $2i \leqslant t$.*

Proof. (i) $\to$ (ii) since V_i is generated by $B^0, \ldots, B^i$.

(ii) $\to$ (iii): This is obvious for $t = 0$, so assume by induction that (iii) holds with $t - 1$ instead of t. If $i + k \leqslant t$, $i, k \leqslant s$, then $E_i \circ E_k = \sum_{l=0}^{\min(i,k)} a_{ik}^l E_l$ for certain numbers a_{ik}^l. If $i < k$, then for $m \leqslant i$, $0 = E_i \circ E_k \circ E_m = \sum_{l=0}^{i} a_{ik}^l (E_l \circ E_m) = a_{ik}^m (\mathrm{const})\, E_m$, so $a_{ik}^m = 0$ for all $m \leqslant i$, and $E_i \circ E_k = E_k \circ E_i = 0$. If $i = k$, then in the same way $a_{ik}^m = 0$ for $m < i$ so that $E_i \circ E_i = a_{ii}^i E_i$. Hence (iii) holds.

(iii) $\to$ (i): By (2.1), $B^i \circ B^k \in V_i \circ V_k = \langle E_{i'} \circ E_{k'} \mid i' \leqslant i,\ k' \leqslant k \rangle = \langle E_{i'} \mid i' \leqslant \min(i, k) \rangle = V_{\min(i,k)}$, and (i) follows.

(iii) $\to$ (iv): Take e_i as before. By (2.2), $1 = j^T(E_i \circ E_i) j = (\mathrm{const})\, j^T E_i j$ for $2i \leqslant t$, whence in this case $e_i \neq 0$, and so $d_i > 0$. Since $D_i = e_i E_i$, we have $D_i \circ D_k = (\mathrm{const})'\, \delta_{ik} D_k$, for $i + k \leqslant t$, $i, k \leqslant s$, and by (2.5) and (2.4), this constant is 1.

We say that B is *t-regular* if the conditions of Theorem 2.4 are satisfied. Obviously, t-regularity implies i-regularity for all $i \leqslant t$.

2.5. THEOREM. *A connected Fiedler matrix B is always 0-regular; it is 1-regular iff the diagonal entries of B are constantly r, and 2-regular iff, in addition, its off-diagonal entries are 0 and $-\lambda$, for some $\lambda > 0$. (Thus we obtain a regular graph on X by calling $x, y \in X$ adjacent iff $b_{xy} = -\lambda$.)*

Proof. 0-regularity requires $I \circ I = (\text{const})I$ which is always true. 1-regularity requires in addition that $B \circ I = (\text{const})I$, i.e., B has constant diagonal. 2-regularity requires in addition that $B \circ B = \mu I - \lambda B$, and $B^2 \circ I = (\text{const})I$. The first equation implies that $b_{xy}^2 = -\lambda b_{xy}$, i.e., $b_{xy} \in \{0, -\lambda\}$ for $x \neq y$. If $b_{xx} = r$, then for given x there are exactly $\lambda^{-1} r$ elements $y \in X$ with $b_{xy} = -\lambda$ (since B has zero row sums). Hence $B^2 \circ I = (r^2 + r\lambda)I$; so the second equation is a consequence of the first.

2.6. LEMMA. *The following conditions are equivalent:*

(i) $D_i \circ D_i = D_i$,

(ii) *D_i is a (0, 1)-matrix,*

(iii) *D_i is an integral matrix.*

Proof. (i) $\leftrightarrow$ (ii) $\rightarrow$ (iii) is obvious. (iii) $\rightarrow$ (ii) follows from (2.4) for $i = k$ which states $\sum d_{ixy}(1 - d_{ixy}) = 0$ where $D_i = (d_{ixy})$.

2.7. THEOREM. *Let B be a connected t-regular Fiedler matrix of geometric rank s, and $t \leqslant 2s$.*

(i) *If $t \geqslant 2e$, then $D_0, \dots, D_e$ are nonzero (0, 1)-matrices, and $d_0, \dots, d_e$ are positive integers.*

(ii) *If $t \geqslant s - 1$, then all polynomials in B have constant diagonal; in particular, $D_1, \dots, D_s$ have zero diagonal.*

(iii) *If $t \geqslant 2s - 2$, then V is closed under pointwise multiplication, and B is $2s$-regular.*

Proof. (i) follows from Theorem 2.4(iv), Lemma 2.6, and (2.5).

(ii) If $t \geqslant s - 1$, then $E_i \circ I = (\text{const})\, E_i \circ E_0 = 0$ for $i = 1, \dots, s-1$. Since $J \circ I = I$, the first assertion follows from Lemma 2.3(iii). Now $D_i \circ I = e_i E_i \circ I = 0$ for $i = 1, \dots, s-1$, and by Lemma 2.3(i) and (2.6),

$$D_s \circ I = \sum_{i=1}^{s} D_i \circ I = (J - I) \circ I = 0.$$

(iii) If $t \geqslant 2s - 2$, then by Theorem 2.4(iii), $E_i \circ E_j \in V$ for $i, j = 0, \dots, s-1$, and obviously $E_i \circ J, J \circ J \in V$. Hence by Lemma 2.3(iii), V

is closed under pointwise multiplication. By (i) and Lemma 2.3(i), $D_0,\dots, D_s$ are matrices whose sum is J, and by (i), $D_0,\dots, D_{s-1}$ are (0, 1)-matrices. Hence D_s is a (0, 1)-matrix, too, and $D_i \circ D_k = \delta_{ik} D_k$. By (i) and Lemma 2.3(ii), all D_i are nonzero, so that $D_i = e_i E_i$ implies Theorem 2.4(iii) with $t = 2s$, i.e., $2s$-regularity.

Remark. Theorem 2.7(iii) implies that V is the adjacency algebra of a P-polynomial association scheme in the sense of Delsarte [4].

3. The Characteristic Matrix

In the following, B is a fixed connected Fiedler matrix. We show that the $p_i(x)$ form a family of orthogonal polynomials, and derive some formulas which allow us to calculate in the B-algebra. These results are relevant, e.g., for the investigation of perfect e-error correcting codes in $2e$-regular graphs. This will be done somewhere else. (For the case $t = 2s - 2$ see, e.g., Delsarte [4].)

We shall always assume that $d_i > 0$ for $i = 0,\dots, s$. By (2.1) and the proof of Theorem 2.2, this is equivalent to the assumption that $D_0,\dots, D_i$ generate V_i.

3.1. Lemma

(i) $D_i D_j = v^{-1} \sum_{\alpha \in S'} p_i(\alpha)\, p_j(\alpha)\, J_\alpha$.
(ii) $\operatorname{tr}(D_i D_j) = j^T(D_i \circ D_j) j = v d_i \delta_{ij}$.
(iii) $J_\alpha = f_\alpha \sum_{i=0}^{s} d_i^{-1} p_i(\alpha) D_i$.
(iv) $D_i D_j = \sum_{l=0}^{s} p_{ij}^l D_l$,

where

$$p_{ij}^l = \frac{1}{v d_l} \sum_{\alpha \in S'} f_\alpha\, p_i(\alpha)\, p_j(\alpha)\, p_l(\alpha). \tag{3.1}$$

Proof. (i) follows from (2.3) and (1.6), and (ii) from (2.4) and (2.5). By the above remark, $J_\alpha = \sum_{i=0}^{s} d_i(\alpha)\, D_i$ for certain numbers $d_i(\alpha)$. Hence $\operatorname{tr}(J_\alpha D_j) = \sum_{i=0}^{s} d_i(\alpha) \operatorname{tr}(D_i D_j) = v d_j d_j(\alpha)$. On the other hand, $\operatorname{tr}(J_\alpha D_j) = \operatorname{tr}(p_j(\alpha) J_\alpha) = v f_\alpha\, p_j(\alpha)$ by (2.3) and (1.6). Hence $d_j(\alpha) = d_j^{-1} f_\alpha\, p_j(\alpha)$, i.e., (iii) holds. (iv) follows by inserting (iii) into (i).

3.2. Lemma (Orthogonality relations)

(i) $\sum_{\alpha \in S'} f_\alpha\, p_i(\alpha)\, p_j(\alpha) = v d_i \delta_{ij}$.

(ii) $\sum_{i=0}^{s} \frac{1}{vd_i} p_i(\alpha)\, p_i(\alpha) = \frac{1}{f_\alpha} \delta_{\alpha\beta}$.

(iii) $p_{ij}^0 = d_i \delta_{ij}$.

(iv) $p_{ij}^l \neq 0$ *if* $|i-j| = l$ *or* $i+j=l$.

(v) $p_{ij}^l \neq 0$ *implies* $|i-j| \leqslant l \leqslant i+j$.

Proof. (i) and (ii) follow by substituting (2.3) and Lemma 3.1(iii) into each other and comparing coefficients. (iii) follows from (1), (2.6), and (i). Lemma 3.1(iv) implies that $p_{ij}^l = 0$ for $l > i+j$, and $\neq 0$ for $l = i+j$, since $D_i D_j \in V_{i+j} \backslash V_{i+j-1}$ for $i+j \leqslant s$. Since by (3.1), $d_l p_{ij}^l$ is symmetric in i, j, and l, this implies (iv) and (v).

The matrix $T = (\tau_{il})$, where for $i, l = 0, \ldots, s$,

$$\tau_{il} = \frac{1}{vd_l} \sum_{\alpha \in S'} \alpha f_\alpha\, p_i(\alpha)\, p_l(\alpha), \tag{3.2}$$

plays an important role and is called the *characteristic matrix* of B. We also define $\tau_{0,-1} = 0$.

3.3. THEOREM

(i) The characteristic matrix $T = (\tau_{il})$ is tridiagonal and satisfies

$$\tau_{il} = 0 \qquad \textit{if} \quad l < i-1, \tag{3.3a}$$

$$\tau_{i,i-1} \neq 0, \tag{3.3b}$$

$$\tau_{ii} = -\tau_{i-1,i} - \tau_{i+1,i}, \tag{3.3c}$$

$$\tau_{i,i+1} \neq 0, \tag{3.3d}$$

$$\tau_{il} = 0 \qquad \textit{if} \quad l > i+1. \tag{3.3e}$$

(ii) d_i *can be recursively defined by*

$$d_0 = 1, \tag{3.4a}$$

$$\tau_{i-1,i} d_i = \tau_{i,i-1} d_{i-1} \qquad \textit{for} \quad i = 1, \ldots, s. \tag{3.4b}$$

(iii) $p_i(x)$ *can be recursively defined by*

$$p_{-1}(x) = 0, \qquad p_0(x) = 1, \tag{3.5a}$$

$$\tau_{i,i+1}\, p_{i+1}(x) = (x - \tau_{ii})\, p_i(x) - \tau_{i,i-1}\, p_{i-1}(x). \tag{3.5b}$$

(iv) *D_i can be recursively defined by*

$$D_{-1}=0, \qquad D_0=1, \tag{3.6a}$$

$$\tau_{i,i+1}D_{i+1}=BD_i-\tau_{ii}D_i-\tau_{i,i-1}D_{i-1}. \tag{3.6b}$$

Proof. By (3.2) and Lemma 3.2(ii), $\sum_{l=0}^{s} \tau_{il}\, p_l(\alpha)$ simplifies to $\alpha p_i(\alpha)$. Hence

$$xp_i(x) \equiv \sum_{l=0}^{s} \tau_{il}\, p_l(x) \qquad \bmod x \operatorname{Ann}_B(x). \tag{3.7}$$

A comparison of the degree shows that $\tau_{il}=0$ if $l>i+1$, and $\tau_{i,i+1}\neq 0$. By (3.2),

$$\tau_{il}d_l=\tau_{li}d_i, \tag{3.8}$$

whence also $\tau_{il}=0$ if $l<i-1$, and $\tau_{i,i-1}\neq 0$. Now $\tau_{l-1,l}+\tau_{ll}+\tau_{l+1,l}=\sum_{i=0}^{s}\tau_{il}=(1/vd_l)\sum_{\alpha\in S'} f_\alpha \alpha(\sum_{i=0}^{s} p_i(\alpha))\, p_l(\alpha)=0$ by Lemma 2.3(i) since $\alpha \operatorname{Ann}_B(\alpha)=0$ for $\alpha\in S'$. This proves (3.3a)–(3.3e). Using (2.6), (3.4) follows now from (3.8), (3.5) from (3.7), and (3.6) from (3.5) and (2.3).

Remarks. (1) Results Lemma 3.2(i, ii) and Theorem 3.3(iii) show that the $p_i(x)$ are a family of orthogonal polynomials, cf. Szegö [7].

(2) The characteristic matrix contains a lot of arithmetical information about B: We can compute the d_l by (3.4), the $p_l(x)$ by (3.5), v and $\operatorname{Ann}_B(x)$ by Lemma 2.3(i), S as the set of zeros of $\operatorname{Ann}_B(x)$, and the f_α by Lemma 3.2(ii).

3.4. Theorem

(i) *The algebra U of polynomials in T is generated by the pairwise orthogonal idempotents*

$$T_\alpha=\left(\frac{f_\alpha}{vd_l}\, p_i(\alpha)\, p_l(\alpha)\right), \qquad \alpha\in S'. \tag{3.9}$$

(ii) *$p(T)=\sum_{\alpha\in S'} p(\alpha)\, T_\alpha$ for all polynomials $p(x)$.*

(iii) *The minimal polynomial of T is a scalar multiple of $x \operatorname{Ann}_B(x)$.*

(iv) *$z_\alpha=(p_i(\alpha))$ is eigenvector of T for the eigenvalue $\alpha\in S'$.*

(v) *The correspondence $T\to B$ induces an isomorphism φ between the algebras U and V. Moreover $\varphi(T_\alpha)=v^{-1}J_\alpha$.*

Proof. By Lemma 3.2(ii), (3.2), and Theorem 3.2(iii), we get easily

$$I = \sum_{\alpha \in S'} T_\alpha, \tag{3.10}$$

$$T = \sum_{\alpha \in S'} \alpha T_\alpha, \tag{3.11}$$

$$T_\alpha T_\beta = \delta_{\alpha\beta} T_\alpha \qquad \text{for} \quad \alpha, \beta \in S'. \tag{3.12}$$

By induction from (3.10)–(3.12), $T^n = \sum_{\alpha \in S'} \alpha^n T_\alpha$, and (ii) follows. By (3.12), the T_α are linearly independent, so (ii) implies (iii). By (ii) and (iii), U is contained in the algebra generated by the T_α and has the same dimension $s+1$, whence (3.1) holds. By (3.7), $Tz_\alpha = \alpha z_\alpha$ which implies (iv). (v) follows from (3.12) and (1.6).

3.5. PROPOSITION. *If B is $2e$-regular, $e \geqslant 1$, then*

$$p_{ij}^l < 0 \qquad \textit{for} \quad i, j, l = 0,..., e, \tag{3.13}$$

$$\tau_{i,i-1} < 0, \qquad \tau_{i-1,i} < 0 \qquad \textit{for} \quad i = 1,..., e. \tag{3.14}$$

Proof. If B is $2e$-regular, then $D_0,..., D_e$ are (0, 1)-matrices whence for $i, j, l = 0,..., e$, $\quad 0 \leqslant \operatorname{tr}(D_i D_j D_l) = v^{-1} \operatorname{tr}(\sum_{\alpha \in S'} p_i(\alpha) p_j(\alpha) p_l(\alpha) J_\alpha) = v d_l p_{ij}^l$ by (2.3), (1.6), (1.7), and (3.1). This implies (3.13). By (2.2), $B \in \langle D_0, D_1 \rangle$ with $D_0 = I$, and D_1 a (0, 1)-matrix, so Theorem 2.4(iii) implies that $B = rD_0 - \lambda D_1$, i.e., $x = rp_0(x) - \lambda p_1(x)$, whence $\tau'_{i,i-1} = rp_{i0}^{i-1} - \lambda p_{i1}^{i-1} = -p_{i1}^{i-1} < 0$, by (3.1), (3.2), (3.13), and Lemma 3.2(iv). Equation (3.14) follows now from (3.4b).

Remarks. (1) By (3.1), (3.9), and Theorem 3.4(ii), p_{ij}^l is the (i, l)-entry of $p_j(T)$.

(2) If B is 2-regular, then, in the notation of Theorem 2.5, $p_1(x) = (r - x)/\lambda$, $d_1 = r/\lambda$.

In the next section we shall need the following result which holds without the assumption that all d_i are positive.

3.6. LEMMA. *Let B be a t-regular Fiedler matrix, and $e = \lfloor t/2 \rfloor$. Then $D_i D_j \circ D_l = p_{ij}^l D_l$ for all $i, j, l = 0,..., e$ satisfying $i + j + l \leqslant t$. Moreover,* (3.3d) *and* (3.6b) *hold for $i = 0,..., e-1$.*

Proof. Define the polynomial $\bar{p}_i(x)$ by $E_i = \bar{p}_i(B)$, so that $D_i = e_i E_i$, $p_i(x) = e_i \bar{p}_i(x)$, $d_i v = e_i^2$, and by Theorem 2.7(i), $d_i e_i \neq 0$ for $i = 0,..., e$.

Essentially the same proofs as for Lemmas 3.1 and 3.2 and Theorem 3.3 show that

$$E_i E_j = \sum_{l=0}^{s} \bar{p}_{ij}^l E_l, \tag{3.15}$$

$$\bar{p}_{ij}^l \neq 0 \qquad \text{implies} \quad l \leqslant i+j, \tag{3.16}$$

where $\bar{p}_{ij}^l = \sum_{\alpha \in S'} f_\alpha \, \bar{p}_i(\alpha) \, \bar{p}_j(\alpha) \, \bar{p}_l(\alpha)$; and

$$\bar{\tau}_{l,l+1} E_{l+1} = BE_l - \bar{\tau}_{ll} E_l - \bar{\tau}_{l,l-1} E_{l-1}, \tag{3.17}$$

where

$$\bar{\tau}_{ll} = \sum_{\alpha \in S'} f_\alpha \, \bar{p}_l(\alpha) \, \bar{p}_l(\alpha), \tag{3.18}$$

$$\bar{\tau}_{l,l+1} \neq 0. \tag{3.19}$$

By (3.15), (3.16), and Theorem 2.4(iii, iv), we have for i, j, l in the required range, $E_i E_j \circ E_l = \sum_{m=0}^{\min(s,i+j)} \bar{p}_{ij}^m E_m \circ E_l = \bar{p}_{ij}^l E_l \circ E_l$, whence $D_i D_j \circ D_l = e_i e_j e_l (E_i E_j \circ E_l) = e_l^{-1} e_i e_j \bar{p}_{ij}^l (D_l \circ D_l) = p_{ij}^l D_l$. Finally, (3.3d) and (3.6b) follow from (3.17)–(3.19).

4. t-Regular Graphs

All graphs considered are nonempty, undirected, without loops or multiple edges. The *valency* of a vertex x is the number of vertices adjacent to x. $d(x, y)$ denotes the distance of two vertices x and y, and $p_{ij}(x, y)$ denotes the number of vertices z such that $d(x, z) = i$ and $d(y, z) = j$. The *diameter* of a grah is the largest occurring distance.

We consider the following conditions:

(i) Every vertex is adjacent to exactly k other vertices.

(ii) The number of vertices adjacent to any two adjacent vertices is λ.

(iii) The number of vertices adjacent to any two nonadjacent vertices is μ.

(π_{ij}^l) If $d(x, y) = l$, then $p_{ij}(x, y) = p_{ij}^l$.

A graph Γ is called *regular*, *edge-regular*, resp. *strongly regular* if (i), (i) and (ii), resp. (i)–(iii) hold, and *distance regular* if Γ is connected, and (π_{ij}^l) holds for all $i, j, l = 0, \ldots, \operatorname{diam} \Gamma$. It is easy to see that a distance regular graph is edge-regular, and the distance regular graphs of diameter 2 are just the connected strongly regular graphs. In his book, Biggs [1] deals extensively with regular and distance regular graphs and proves many special cases of the results of Sections 1–3, interpreted in terms of graphs.

For a graph Γ of diameter δ, with vertex set X of size v, we define symmetric $v \times v$-matrices $A_i = (a_{ixy})_{x,y \in X}$, $i = 0,\dots,\delta$, and $B = (b_{xy})_{x,y \in X}$ by

$$\begin{aligned} a_{ixy} &= 1 \quad \text{if} \quad d(x,y) = i, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

$$\begin{aligned} b_{xy} &= \text{valency of } x && \text{if } x = y, \\ &= -1 && \text{if } x \text{ adjacent } y, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then $A_0 = I, A_1,\dots, A_\delta$ are $(0, 1)$-matrices, and B is a Fiedler matrix. We call A_1 the *adjacency matrix* of Γ, and B the *Fiedler matrix* of Γ (cf. Fiedler [8]). We call Γ *t-regular* of *rank* $s+1$ if the Fiedler matrix of Γ is t-regular and has geometric rank s.

4.1. THEOREM. *If Γ is a connected graph with diameter δ and rank $s+1$, then $s \geqslant \delta$.*

Proof. Let a_i, b_i be vertices of distance i, $i = 0,\dots,\delta$. Then the (a_i, b_i)-entry of B is 0 if $l < i$, and $\neq 0$ if $l = i$. Hence if $c_0 B^0 + \cdots + c_\delta B^\delta = 0$, then, considering the (a_i, b_l)-entry for $i = \delta, \delta - 1,\dots, 0$, we find $c_\delta = c_{\delta-1} = \cdots = c_0 = 0$. Hence the minimal polynomial $x \operatorname{Ann}_B(x)$ has degree $s + 1 > \delta$. Therefore $s \geqslant \delta$.

Problem. Characterize the case $s = \delta$.

4.2. PROPOSITION. *Γ is 1-regular iff Γ is 2-regular iff Γ is regular. In this case,*

$$D_1 = A_1 = kI - B. \tag{4.1}$$

Proof. By Theorem 2.5 and Remark (2) after Lemma 3.5.

4.3. THEOREM. *Let Γ be a t-regular graph, and $e = [t/2]$. Then*

(i) *$D_i = A_i$ for $i = 0,\dots, e$,*

(ii) *(π_{ij}^l) holds for all $i, j, l = 0,\dots, e$ satisfying $i + j + l \leqslant t$, with p_{ij}^l as in Lemma* 3.1(iv).

Proof. Of course, $D_0 = I = A_0$. Hence assume that $e \geqslant 1$, and $D_l = A_l$ for $i = 0,\dots, c$, where $c \in \{0,\dots, e-1\}$. By Theorem 2.7, D_{c+1} is a $(0, 1)$-matrix, and by Theorem 2.4(iv) and our assumptions, $D_{c+1} \circ A_i = 0$ for $i \leqslant c$. Now by Lemma 3.6 and Proposition 4.2,

$$-\tau_{c,c+1} D_{c+1} = A_1 A_c + (\tau_{cc} - k) A_c + \tau_{c,c-1} A_{c-1}. \tag{4.2}$$

$A_1 A_c$ has nonzero entries just at places (x, y) with $c - 1 \leqslant d(x, y) \leqslant c + 1$, so $\tau_{c,c+1} \neq 0$ implies that D_{c+1} has nonzero entries just at places (x, y) with $d(x, y) = c + 1$. Hence $D_{c+1} = A_{c+1}$, and (i) follows by induction. To prove (ii), we simply observe that the (x, y)-entry of $A_i A_j$ is $p_{ij}(x, y)$, and by (i) and Lemma 3.6, we have $A_i A_j \circ A_l = p_{ij}^l A_l$ for i, j, l in the required range.

4.4 THEOREM. *Let Γ be a graph of diameter δ, and $e \leqslant \delta - 1$.*

(i) *If (π_{1i}^l) holds for $i = 0, \dots, e-1$ and $l = i-1, i, i+1$, and (π_{ie}^{e-i}) holds for $i = 0, \dots, e$, then Γ is 2e-regular.*

(ii) *If Γ is 2e-regular and satisfies (π_{ie}^{e+1-i}) for $i = 0, \dots, e$, then Γ is $(2e+1)$-regular.*

Proof. Note first that (π_{ij}^l) is trivially satisfied (with $p_{ij}^l = 0$) unless $|i - j| \leqslant l \leqslant i + j$. Hence $A_i A_j = (p_{ij}(x, y))$ implies, together with the assumptions of (i), that for $i = 0, \dots, e$,

$$A_1 A_i = p_{1i}^{i-1} A_{i-1} + p_{1i}^i A_i + p_{1i}^{i+1} A_{i+1} \qquad \text{if} \quad i \neq e, \tag{4.3}$$

$$A_l \circ A_i A_e = 0 \qquad \text{if} \quad l < e - i, \tag{4.4}$$

$$A_{e-i} \circ A_i A_e = p_{ie}^{e-i} A_{e-i}, \tag{4.5}$$

and $p_{1i}^{i+1}, p_{ie}^{e-i} \neq 0$. (π_{11}^0) implies that Γ is regular, whence by Proposition 4.2 and (4.3), A_i is a polynomial of degree i in B (induction for $i = 0, \dots, e$), and hence for $i = 0, \dots, e$,

$$V_i = \langle A_0, \dots, A_i \rangle, \tag{4.6}$$

$$V_{i+e} = \langle A_0, \dots, A_e, A_1 A_e, \dots, A_i A_e \rangle. \tag{4.7}$$

The A_i are (0, 1)-matrices whence

$$A_i \circ A_j = \delta_{ij} A_i. \tag{4.8}$$

Now (4.4)–(4.8) imply $V_i \circ V_j \subseteq V_i$ for $i + j \leqslant 2e$, $i \leqslant j$ (and hence $i \leqslant e$), whence by Theorem 2.4(ii), Γ is 2e-regular.

The additional assumption of (ii) implies similarly

$$A_{e+1-i} \circ A_i A_e = p_{ie}^{e+1-i} A_{e+1-i} \tag{4.9}$$

for $i = 0, \dots, e$, whence $V_i \circ V_j \subseteq V_i$ for $i + j = 2e + 1$, $1 \leqslant i \leqslant j$. Hence Γ is $(2e+1)$-regular if we show that $I \circ A_1 A_e A_e = (\text{const}) I$ to cover the case

$i=0$. But a diagonal entry of $A_1 A_e A_e$ is a row sum of $A_1 \circ A_e A_e$, hence constant by (4.9) for $i=e$ since A_1 has constant row sum k.

4.5. COROLLARY. *A graph is t-regular iff, with $e=[t/2]$, condition 4.3(ii) is satisfied. (Here we assume $t \leqslant 2s$.)*

4.6. COROLLARY. *A graph is 3-regular iff it is edge-regular.*

4.7. THEOREM. *Let Γ be a connected graph with diameter δ and rank $s+1$. Then the following conditions are equivalent:*

(i) *Γ is $(2\delta-2)$-regular, and $s=\delta$,*

(ii) *Γ is $(2s-2)$-regular,*

(iii) *Γ is $2s$-regular,*

(iv) *Γ is distance regular.*

Proof. (i) $\rightarrow$ (ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv) by Theorem 2.7(iii) and Corollary 4.5. (iii) $\rightarrow$ (i): By Theorem 4.3(i), A_i is a polynomial in B of degree i. Moreover, $J=A_0+\cdots+A_\delta$, and $BJ=0$. Hence the minimal polynomial of B has degree $\leqslant \delta+1$, whence by 4.1, $s=\delta$. Hence Γ is 2δ-regular, in particular $(2\delta-2)$-regular.

4.8. COROLLARY. *The strongly regular graphs are just the 2-regular (or 3-regular, or 4-regular) graphs of rank 3.*

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