STRONG BALANCED TUPLE SYSTEMS, 2-DESIGNS, AND NEAR VECTOR SPACES

By A. NEUMAIER

A 2-$(v, k, \lambda; G)$-SBTS (strong balanced tuple system) is a collection Q of k-tuples with distinct entries from a v-set P satisfying

(1) for any two distinct points a and b, and any two distinct places i and j, there are exactly λ tuples $x \in Q$ with $x_i = a$, $x_j = b$,

(2) G is a permutation group on the k places such that for every $\alpha \in G$, the tuple $(x_{\alpha 1}, \ldots, x_{\alpha k})$ is in Q with the same multiplicity as $(x_1, \ldots, x_k)$.

Strong balanced tuple systems have a very close relationship to 2-designs, and occur naturally in some game tournaments.

We study here the properties of strong balanced tuple systems, and, using near vector spaces, derive some large classes of SBTS. From these many 2-designs are derived.

0. Introduction

THE main purpose of this paper is the introduction of a new type of combinatorial structure called a *strong balanced tuple system*. This is at the same time a generalization and a refinement of the concept of a 2-design.

In some sense strong balanced tuple systems are a collection of "situations" satisfying a certain balance condition. A situation may be viewed as a set of points in which the position of a point is fixed up to some symmetries, the remaining asymmetry defining relations between the points. In game tournaments, the relations may be being partner, opponent, or neighbour, the symmetries may be rotating the table, or partners changing places, etc.

The blocks of a design fit into this framework in two ways: either as a situation having all possible symmetries and hence no special relations (in which case strong balanced tuple systems generalize designs), or as the underlying structure of the point sets on which relations are to be imposed in a way such that the balance condition is satisfied (in which case strong balanced tuple systems are designs with refined structure). Putting it another way: a strong balanced tuple system may be viewed either as a design *lacking* symmetry or as a design with *additional* structure.

This additional structure may sometimes help in the construction of a 2-design. In fact, using *near vector spaces* introduced by Quackenbush **[10]** we are able to construct large classes of strong balanced tuple systems, and they give large classes of 2-designs.

To keep the paper self-contained we start with a short introduction to some simple results on 2-designs.

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1. Designs

Let P be a finite set of elements (*points*), and $\mathcal{B}$ be a collection of not necessarily distinct subsets of P (*blocks*) such that every block contains k points, and any two distinct points are in λ blocks. If v is the total number of points, $\mathcal{B}$ is called a 2-(v, k, λ)-*design* (on P).

Well-known counting arguments give $r = \lambda(v-1)/(k-1)$ as the number of blocks containing a given point, and $b = \lambda v(v-1)/k(k-1)$ as the total number of blocks. Hence

$$\lambda v(v-1) \equiv 0 \bmod k(k-1), \tag{1a}$$

$$\lambda(v-1) \equiv 0 \bmod k-1 \tag{1b}$$

are necessary conditions for the existence of a 2-(v, k, λ)-design. Wilson ([**11**], [**12**], [**13**]) proved that conditions (1) are also sufficient for large v, i.e. if $v > v_0(k, \lambda)$ with a constant v_0 depending on k and λ. Another well-known condition for 2-(v, k, λ)-designs with $v > k$, $\lambda > 0$ is the Fisher inequality $b \geqslant v$, or

$$k(k-1) \leqslant \lambda(v-1). \tag{2}$$

An *automorphism* of a design is a permutation π of the points such that for every block B, the set $\{\pi a \mid a \in B\}$ is in the design with the same multiplicity as B.

Good accounts of designs are contained in Hall [**3**], Dembowski [**2**], and Hanani [**4**].

2. Strong balanced tuple systems

A *tuple system* is a triple (P, I, Q) consisting of a v-set P of *points*, a k-set I of *places*, and a collection Q of not necessarily distinct k-tuples with entries from P, the entries being indexed by the elements of I; i.e. the elements of Q are of the form $x = (x_1, \ldots, x_k) = (x_i\colon\ i \in I)$. Usually we refer to Q as a tuple system, mentioning P, I only where necessary. A tuple system is called *strong* if each of its tuples contains no entry twice.

An *automorphism* of a tuple system Q is a pair (α, π) consisting of a permutation α of the places and a permutation π of the points such that for every tuple $x \in Q$, the tuple $(\alpha, \pi)x = (\pi x_{\alpha i}\colon\ i \in I)$ is in Q with the same multiplicity as x. We call an automorphism *free* if it is of the form $\alpha = (\alpha, 1)$, and *pure* if it is of the form $\pi = (1, \pi)$. Free automorphisms play an important role in our theory.

Let us denote the number of tuples $x \in Q$ with $x_i = a$, $x_j = b$ by $\lambda_{ij}(a, b)$. We define a *strong balanced tuple system* (SBTS) as a strong tuple system Q satisfing the axiom

(*) $\lambda_{ij}(a, b) = \lambda$ for any two distinct points a, b, and any two distinct places i, j.

If P has v points, I has k places, and G is a group of free automorphisms of a strong balanced tuple system (P, I, Q) then we call Q a 2-$(v, k, \lambda; G)$-*SBTS*. A simple counting argument gives $\lambda(v-1)$ as the number of $x \in Q$ with $x_i = a$, and $\lambda v(v-1)$ as the total number of tuples in Q.

EXAMPLE 1. Consider a tennis tournament on a single tennis court. We are looking for an arrangement in which each person plays with each other person once as partner and twice as opponent (Since there are twice as many opponents as partners this requirement is natural). The relationship of being partner or opponent is preserved under the dihedral group D_4 generated by (12), (34), and (13)(24). If person x_i is playing on place i such that x_1, x_2 and x_3, x_4 are partners, the other pairs x_i, x_j are opponents then we denote this situation by $(x_1, x_2, x_3, x_4 \mid D_4)$ meaning that, for the balance requirement, the ordering is essential only up to permutations from D_4. For example, the unique solution for 4 persons a, b, c, d is $(abcd \mid D_4)$, $(acdb \mid D_4)$, $(adbc \mid D_4)$. If we associate with every situation $(x_1, x_2, x_3, x_4 \mid D_4)$ the eight tuples $(x_{\alpha 1}, x_{\alpha 2}, x_{\alpha 3}, x_{\alpha 4})$, $\alpha \in D_4$, then the resulting tuple system Q is strong, has D_4 as a group of free automorphisms, the balance condition means simply that $(*)$ holds with $\lambda = 2$, and hence the problem considered is equivalent to the construction of a 2-$(v, 4, 2; D_4)$-SBTS. The given solution in the case $v = 4$ corresponds to the tuple system containing all tuples with distinct entries exactly once.

Note that similar arrangements occur in whist tournaments and have been considered by Moore [7].

EXAMPLE 2. Similarly, we may consider a k-set $\{x_1, \ldots, x_k\}$ as a situation $(x_1, \ldots, x_k \mid S_k)$ "up to arbitrary permutations of places"; and, associating with every k-set $\{x_1, \ldots, x_k\}$ of a 2-(v, k, λ)-design the $k!$ tuples $(x_{\alpha 1}, \ldots, x_{\alpha k})$, $\alpha \in S_k$, we obtain a 2-$(v, k, \lambda(k-2)!; S_k)$-SBTS. For, given two distinct points a and b, there are just λ blocks containing a and b; the positions of a and b are determined by i and j, and the remaining $(k-2)$ points may be distributed arbitrarily over $(k-2)$ places in $(k-2)!$ ways.

EXAMPLE 3. Let P be an abelian group of odd order v, and $I = \{1, 2, 3\}$. Define $Q = \{(a+b, a-b, a) \mid a, b \in P, b \neq 0\}$. Since the equation $2x = a$ has a unique solution, two entries in a triple of Q determine uniquely the third, and (12) is a free automorphism. Hence Q is a 2-$(v, 3, 1; \langle(12)\rangle)$-SBTS. If P is elementary abelian of exponent 3 (i.e. $v = 3^n$) then (123) can be easily shown to be also a free automorphism. Since (12) and (123) generate S_3, we have then a 2-$(3^n, 3, 1; S_3)$-SBTS.

EXAMPLE 4. Mendelsohn [6] considered 2-$(v, 3, 1; Z_3)$-SBTSs with the cyclic group Z_3 of order 3 under the name of *cyclic triple systems*.

There are some divisibility conditions for the parameters of a SBTS which are similar to the conditions (1) for designs. They depend on the orders of stabilizers of the free automorphism group G. Here the stabilizer G_i (G_{ij}) of a place i (of distinct places i and j) is the subgroup of G consisting of all $\alpha \in G$ fixing i (resp. i and j).

PROPOSITION 1. *If a 2-(v, k, λ; G)-SBTS exists, then, for any two distinct places i, j,*

$$\lambda \equiv 0 \bmod |G_{ij}|, \tag{3a}$$

$$\lambda(v-1) \equiv 0 \bmod |G_i|, \tag{3b}$$

$$\lambda v(v-1) \equiv 0 \bmod |G|. \tag{3c}$$

Proof. Given $a \neq b$, $i \neq j$, the collection $Q_1 = \{x \in Q \mid x_i = a, x_j = b\}$ consists of full orbits of G_{ij}. Since Q is strong, two permutations α, β of the places satisfy $\alpha x = \beta x$ iff $\alpha = \beta$; in particular, all the orbits of G_{ij} have length $|G_{ij}|$, and $|G_{ij}|$ divides $|Q_1| = \lambda$. Similarly, $Q_2 = \{x \in Q \mid x_i = a\}$ consists of full orbits of G_i whence $|G_i|$ divides $|Q_2| = \lambda(v-1)$. Finally, Q consists of orbits of G whence $|G|$ divides $|Q| = \lambda v(v-1)$.

It is known (Wilson [**14**]) that conditions (3) are sufficient for the existence of a 2-(v, k, λ; G)-SBTS if v is greater than a constant depending on k, λ, and G. The proof is based on a recursive construction using pairwise balanced designs, and deep methods developed by Wilson ([**11**], [**12**], [**13**]). Neumaier [**8**] proved that conditions (3) are sufficient if $k = 3$, $\lambda = 1$ and $(v; G) \neq (6; Z_3)$. The construction of SBTSs with $k = 4$ raises already many interesting problems.

3. SBTS and designs

Examples 1 and 2 are special cases of general constructions which generate all strong balanced tuple systems. The first example generalizes as follows: Consider a strong tuple system Q of k-tuples with entries from a set of v points, with a group G of free automorphisms. We call a G-orbit of tuples of Q a *situation*, and denote the G-orbit Gx of $x = (x_1, \ldots, x_k)$ by $(x_1, \ldots, x_k \mid G)$. Suppose that G has s orbits (numbered with the first s positive integers) on the pairs (i, j) of distinct places. Call two places *nth associates* if the corresponding pair is in the nth G-orbit (cf. Higman [**5**] for the resulting "coherent configuration"). If, in a situation $(x_1, \ldots, x_k \mid G)$, i and j are nth associates then we say that x_i and x_j are nth associates in that situation. (Note that the relation of being nth associates need not be symmetric).

THEOREM 1. (i) *If $v > 1$, and Q satisfies*

(**) *Given two distinct points a, b, and an integer n, $1 \leq n \leq s$, there are exactly λ_n situations in which a and b are nth associates,* *then, for any pair (i, j) of nth associates, $\lambda = \lambda_n |G_{ij}|$ is independent of n and (i, j), and Q is a 2-$(v, k, \lambda; G)$-SBTS.* (ii) *If Q is a 2-$(v, k, \lambda; G)$-SBTS with $v > 1$ then (**) is satisfied with $\lambda_n = |G_{ij}|^{-1} \cdot \lambda$, where (i, j) are nth associates.*

Remark. In our Example 1 we have $s = 2$, partners are first associates, opponents are second associates, and (**) is satisfied with $\lambda_1 = 1$, $\lambda_2 = 2$.

Proof. (i) Suppose that (**) is valid. Given two distinct points a and b, and a pair (i, j) of nth associate places, there are λ_n situations in which a and b are nth associates. To every situation there are exactly $|G|$ corresponding tuples, of which exactly $|G_{ij}|$ have $x_i = a$, $x_j = b$. Hence $\lambda_{ij}(a, b) = \lambda_n |G_{ij}|$, independently of a and b. Therefore, the total number of tuples is the sum over the $v(v-1)$ numbers $\lambda_{ij}(a, b)$, $a \neq b$, which is $v(v-1)\lambda_n |G_{ij}|$. This is independent of n and (i, j), whence also $\lambda = \lambda_n |G_{ij}|$ is independent of n and (i, j). Hence $\lambda_{ij}(a, b) = \lambda$ for all $a \neq b$, $i \neq j$, and Q is a 2-$(v, k, \lambda; G)$-SBTS. (ii) Conversely, suppose that Q is a 2-$(v, k, \lambda; G)$-SBTS. Take two distinct points a and b, and an integer n, $1 \leq n \leq s$. If i and j are nth associate places, there are exactly λ tuples $x \in Q$ with $x_i = a$, $x_j = b$. The number of pairs of nth associate places is $(G: G_{ij})$, where i and j are nth associates. Hence there are $\lambda(G: G_{ij})$ tuples $x \in Q$ which have a and b on nth associate places. These tuples decompose into G-orbits of length $|G|$ each, i.e. there are exactly $|G|^{-1}\lambda(G: G_{ij}) = |G_{ij}|^{-1}\lambda$ situations in which a and b are nth associates. Hence (**) is valid with $\lambda_n = |G_{ij}|^{-1}\lambda$.

THEOREM 2. *Let G be a 2-transitive group on k places. Then a 2-(v, k, λ_1)-design exists iff a 2-$(v, k, \lambda; G)$-SBTS exists with $\lambda = \lambda_1 |G|/k(k-1)$.*

Proof. Let $\mathcal{B}$ be a collection of k-sets. We label the k entries of each block arbitrarily as $x_1, \ldots, x_k$, form the situation $(x_1, \ldots, x_k \mid G)$, and put all the tuples belonging to this situation inito a tuple system Q. Conversely, given Q, put into $\mathcal{B}$ all sets $\{x_1, \ldots, x_k\}$ where $(x_1, \ldots, x_k \mid G)$ is a situation of Q. Now G is 2-transitive iff $s = 1$, i.e. there is only one associate class, and $|G| = k(k-1)|G_{ij}|$. Hence (**) is equivalent to the statement that for any two distinct points there are exactly λ_1 situations in which a and b occur. Hence, by Theorem 1, $\mathcal{B}$ is a 2-(v, k, λ_1)-design iff Q is a 2-$(v, k, \lambda; G)$-SBTS. Note that the SBTS determines uniquely the design, but the design determines the SBTS only if $G = S_k$, the symmetric group, since otherwise the points in a situation may be arranged in different ways.

The connection between strong balanced tuple systems and designs is still more closely:

THEOREM 3. *Even if G is not 2-transitive, the existence of a*

2-$(v, k, \lambda; G)$-SBTS implies the existence of a 2-(v, k, λ_1)-design with $\lambda_1 = |G|^{-1}k(k-1)$.

Proof. Let Q be a 2-$(v, k, \lambda; G)$-SBTS. We associate with every situation $(x_1, \ldots, x_k \mid G)$ the set $\{x_1, \ldots, x_k\}$ and claim that the collection of these sets is the required 2-design. In fact, if a and b are distinct points then a and b lie in the block $\{x_1, \ldots, x_k\}$ iff there are distinct places i, j such that $x_i = a$, $x_j = b$. There are $k(k-1)$ choices for i and j, λ possibilities for x when i, j are given, and hence $\lambda k(k-1)$ possible tuples x. But the $|G|$ tuples in the same G-orbit determine the same block whence the number of blocks containing a and b is $|G|^{-1}$ times this number.

We say that a 2-(v, k, λ_1)-design $\mathcal{B}$ *admits a group* G if there is a 2-$(v, k, \lambda; G)$-SBTS Q with $\lambda \neq \lambda_1 |G|/k(k-1)$ such that the construction of Theorem 3 produces $\mathcal{B}$ from Q. By Theorem 1, this means essentially that the blocks may be ordered (with the partial symmetry induced by G) such that the stronger balance condition (**) is satisfied. We proved in Theorem 2 that a design certainly admts a group G if G is large enough (2-transitive, or even, trivially, the symmetric group); hence it is an interesting question when a design admits a small group. This is not always possible, not even if the necessary conditions (3) are satisfied; e.g. for the cyclic group Z_3 of order 3 there is a 2-$(6, 3, 2)$-design but no 2-$(6, 3, 1; Z_3)$-SBTS (A proof by exhaustion is quite easy). Some large classes of designs which admit small groups are given in section 6.

A Hadamard 2-$(4n-1, 2n-1, n-1)$-design might admit a dihedral group D_{2n-1} yielding a 2-$(4n-1, 2n-1, 1; D_{2n-1})$-SBTS. Such SBTSs exist if $4n-1$ is a prime power (see the corollary to Theorem 5). I do not know of other cases; a solution would imply (see Neumaier [**8**]) a set of $2n-3$ mutually orthogonal Latin squares of side $4n-1$ (a number presently known to be attained only for prime powers $4n-1$).

Theorem 3 has a nice corollary which bounds the number of free automorphisms of a SBTS:

COROLLARY. *If Q is a 2-$(v, k, \lambda; G)$-SBTS with $v > 1, k > 1, \lambda > 0$ and $|G| > \lambda(v-1)$ then $v = k$.*

Proof. Since $v > 1$ there are two distinct points a and b, and $\lambda > 0$ implies then the existence of a tuple in Q; hence $v \geq k$ since Q is strong. Now suppose that $v > k$. Then the induced design of Theorem 3 has $v > k$ and $\lambda_1 > 0$. Hence the Fisher inequality (2) applies, giving $|G| \leq \lambda(v-1)$. This contradicts the hypothesis whence $v = k$.

The case $v = k$, $\lambda = 1$ is equivalent to an affine plane with a group G^* of dilatations fixing an affine line, such that the operation of G^* on that line is permutation equivalent to the action of G on the places (see Neumaier [**9**]). Nothing is known about the case $v = k$, $\lambda > 1$.

The three theorems show clearly the importance of free automorphisms of a SBTS; they influence the structure of a SBTS in a crucial way. There has not been done any work on general automorphisms. But it is worth to notice that every non-free automorphism (α, π) of a SBTS induces the automorphism π in the design constructed in Theorem 3. It would be interesting to know when all automorphisms of the induced design arise in this way. Conversely, if G is the symmentric group then the SBTS corresponding (by Theorem 2) to a design has the automorphisms of the design as pure automorphisms. If G is only 2-transitive then the automorphisms of the corresponding SBTS depend on the chosen arrangement of the points in the situations; it is not known whether the ordering may be chosen in such a way that some (or all) automorphisms may be lifted to (pure?) automorphisms of the SBTS.

4. Near modules and near vector spaces

A set R together with a binary operation $+$ on R is a *loop* if R contains an element 0 satisfying $x+0=0+x=x$, and if for given $a, b \in R$ there are unique elements $x, y \in R$ with $x+a=b$, $a+y=b$. For the theory of loops see Bruck [**1**]. A *commutative, diassociative loop* is a loop such that any two elements of R generate an abelian group. The *exponent* of a loop is the smallest positive integer n such that $x^n = 1$ for all x.

Let K be a (commutative, associative) ring with unity 1. A *K-nearmodule* is a loop $(R, +)$ with K as a left operator ring satisfying

(N1) $a(x+y) = ax+ay$,
(N2) $ax+(bx+y) = (a+b)x+y$,
(N3) $(ab)x = a(bx)$,
(N4) $1x = x$,

for every $a, b \in K$, $x, y \in R$. If K is a field then we call R a *near vector space* over K. In this case (N1) and (N2) imply that the K-nearmodule generated by two elements of R is a vector space of dimension ≤ 2 over K.

Examples of commutative, diassociative loops are abelian groups, and commutative Moufang loops (see Bruck [**1**]). Examples of near modules are ordinary K-modules, and commutative di-associative loops of exponent n, with K being the integers mod n. Near vector spaces can be found by a method of Quackenbush [**10**], see Theorem Q below.

Similarly as in the construction of projective geometries from vector spaces we obtain from any near vector space R over K a 2-design by calling points the 1-dimensional subspaces Kx $(x \neq 0)$, and blocks the set of 2-dimensional subspaces $Kx+Ky$ $(x \neq 0, y \notin Kx)$. If $|R| = v$, $|K| = q$ then the resulting design has parameters $2\text{-}((v-1)/(q-1), q+1, 1)$. In particular, $v \equiv 1 \bmod (q-1)$. Although nonisomorphic near vector spaces may produce isomorphic designs, the construction has a converse:

THEOREM Q (Quackenbush [**10**]). *Let q be a prime power, and $K = GF(q)$ be the field with q elements. Then a near vector space over K with v elements exists iff there is a 2-$((v-1)/(q-1), q+1, 1)$-design.*

5. SBTS from near modules and near vector spaces

PROPOSITION 2. *Let K be a ring with unity, and $K^\times$ be the group of units of K. Let R be a finite K-near module with v elements. Suppose that I is a k-subset of K such that*

$$i - j \in K^\times \quad \text{for every } 1, j \in I, \quad i \neq j. \tag{4}$$

Then $Q = \{(ix + y\colon\ i \in I) \mid x, y \in R, x \neq 0\}$ is a 2-$(v, k, 1)$-SBTS. Moreover, if the permutation $i \mapsto \varepsilon i + a$ $(a \in K, \varepsilon \in K^\times)$ preserves I then it is a free automorphism of Q.

Proof. If $(i-j) \in K^\times$ then $ix + y = a$, $jx + y = b$ have the unique solution $x = (i-j)^{-1}(a-b)$, $y = a - ix$. From this it follows easily that Q is a 2-$(v, k, 1)$-SBTS. The permutation $\alpha\colon\ i \mapsto \varepsilon i + a$ transforms the tuple $[x, y] := (ix + y\colon\ i \in I)$ into $((\varepsilon i + a)x + y\colon\ i \in I) = (i\varepsilon x + ax + y\colon\ i \in I) = [\varepsilon x, ax + y]$, and both tuples occur in Q with multiplicity one. Hence if α preserves I then it is a free automorphism of Q.

For example, if we take for K the integers module an odd number n, then $I = \{0, 1, -1\}$ satisfies (4) and is preserved by $i \mapsto -i$. Hence Q is a 2-$(v, 3, 1; \langle(1, -1)\rangle)$-SBTS. If R is an abelian group of odd order then this is the SBTS given in section 3, Example 3; but now we see that we get a SBTS in the same way from any commutative, disassociative loop of odd exponent.

Since it is not easy, in general, to deal with condition (4), we now specialize to the case when K is a field, and R is a near vector space over K. In this case, (4) is satisfied for all subsets I of K, which makes everything easier. We shall construct SBTS with the following groups of free automorphisms:

$A_k(q^t, d)$ = the split product of an elementary abelian group V of order q^t acting semiregularly on the places, and a cyclic group Z of order d acting semiregularly on the orbits of V;

$A'_k(q^t, d)$ = the split product of an elementary abelian group V of order q^t acting semiregularly on the places, and a cyclic group Z of order d fixing one orbit of V and acting semiregularly on the remaining orbits;

$Z_k(d)$ = a cyclic group of order d acting semiregularly on the places:

$Z'_k(d)$ = a cyclic group of order d fixing a place and acting semiregularly on the remaining places;

$A(q)$ = the (2-transitive) group of all integral linear transformations of a field with q elements;

D_k = the dihedral group of degree k and order $2k$.

(The split product of an abelian group V and a group Z of automorphisms of V is the group of transformations of V of the form $x \mapsto \varepsilon x + a$, $a \in V$, $\varepsilon \in Z$. A group acts semiregularly on a set X if only the identity fixes an element of X.)

PROPOSITION 3. *Let q be a prime power, and let $K = GF(q^s)$ be the field with q^s elements. In the following cases there is a k-subset I of K which is preserved by a linear transformation group isomorphic to G:*

(a) $G = A_k(q^t, d)$ *and* $d \mid q-1$, $k \equiv 0 \bmod q^t d$, $k \leq q^s - q^t$, $t \leq s$;
(b) $G = A_k^!(q^t, d)$ and $d \mid q-1$, $k \equiv q^t \bmod q^t d$, $k \leq q^s$, $t \leq s$;
(c) $G = Z_k(d)$ and $d \mid q-1$, $d \mid k \leq q^s - 1$;
(d) $G = Z_k^!(d)$ and $d \mid q-1$, $d \mid k-1 \leq q^s - 1$;
(e) $G = A(q)$ and $k = q$.

Proof. K contains a subfield $K = GF(q)$. Since $d \mid q-1$, K contains a cyclic subgroup H of order d. Let K_0 be a vector space of dimension t over K contained in K.

(a) Since $k \equiv 0 \bmod q^t d$ we may take for I the union of some sets of the form $K_0 + Hb$ $(b \in K - K_0)$. The transformations $i \mapsto \varepsilon i + a$ $(a \in K_0, \varepsilon \in H)$ preserve I and form an $A_k(q^t, d)$.

(b) The same transformations preserve also K_0, whence they preserve $I' = I \cup K_0$ and act on it as $A_k^!(q^t, d)$.

(c) and (d) are the special cases $t = 0$ of (a) and (b), and (e) is the special case $t = 1$, $k = q$, $d = q-1$ of (b).

A combination of the two propositions gives

THEOREM 4. *Let q be a prime power, $K = GF(q^s)$, and R be a finite near vector space over K with v elements. Then, in the cases (a)–(e) of above, a 2-$(v, k, 1; G)$-SBTS exists.*

To obtain another class of SBTS we use slightly varied version of Proposition 2 to prove

THEOREM 5. *Let R be a near vector space over $K = GF(q)$ with v elements. Then a 2-$(v, k, 1; D_k)$-SBTs exists when*

$$k \mid q-1, \quad k \text{ odd}, \quad q \text{ odd}. \tag{5}$$

Proof. Let α be a primitive k-th root of unity in K. By (5) the elements $\pm 1, \pm\alpha, \ldots, \pm\alpha^{k-1}$ are distinct and form a subgroup H of $K^\times$. Choose a set S_0 of representatives of the orbits of H on $R - \{0\}$ and define $S = S_0 \cup \alpha S_0 \cup \cdots \cup \alpha^{k-1} S_0$. S has the properties

$$\alpha S = S, \quad S \cap -S = \varnothing, \quad S \cup -S = R - \{0\}. \tag{6}$$

With any set S satisfying (6) we form

$$Q=\{(\alpha^{\varepsilon i}x+y:\ i \bmod k) \mid x\in S,\ y\in R,\ \varepsilon=\pm 1\}.$$

Clearly Q is strong. To show that Q is a SBTS we have to prove that $\lambda_{ij}(a,b)=1$ if $a\neq b$, $i\neq j$. Now $\lambda_{ij}(a,b)$ is the number of solutions of

$$\alpha^{\varepsilon i}x+y=a;\quad \alpha^{\varepsilon i}x+y=b. \tag{7}$$

These imply

$$x=(\alpha^{\varepsilon i}-\alpha^{\varepsilon j})^{-1}(a-b). \tag{8}$$

Now $\alpha^{-i}-\alpha^{-j}=-\alpha^{-i-j}(\alpha^{i}-\alpha^{j})$ whence by (6) the expression (8) is in S for exactly one choice of $\varepsilon=\pm 1$. Therefore ε, x, and (by (7)) y are uniquely determined by (7). Hence Q is SBTS. The transformations $i\mapsto \alpha i$ and $i\mapsto i^{-1}$ are free automorphisms of Q and generate D_k.

If we take $R=K$ in Theorem 5 we obtain the

COROLLARY. *A 2-$(v, k, 1; D_k)$-SBTS exists whenever k is an odd integer and v is a prime power with $v\equiv 1 \bmod 2k$.*

6. Some classes of 2-designs

THEOREM 6. *Let q be a prime power, and suppose that a 2-$(w, q^s+1, 1)$-design exists. Then a 2-$(w(q^s-1)+1, k, \lambda)$-design exists in each of the following cases:*

(a) $q^t \mid k\leq q^s-q^t,\ k-1\mid \lambda,\ k(k-1)\mid \lambda q^t(q-1),\ t\leq s$;
(b) $q^t \mid k\leq q^s,\ k\mid q^t\lambda,\ k(k-1)\mid \lambda q^t(q-1),\ t\leq s$;
(c) $k\leq q^s-1,\ k-1\mid\lambda,\ k(k-1)\mid \lambda(q-1)$;
(d) $k\leq q^s,\ k\mid\lambda,\ k(k-1)\mid\lambda(q-1)$;
(e) $k=q,\ \lambda=1$;
(f) $k\mid q^s-1,\ \lambda=\frac{1}{2}(k-1)$, k odd, q odd.

In particular, in these cases a 2-(q^s, k, λ)-design exists.

Proof. (a)–(e) follow from Theorem 4, and (f) from Theorem 5 (with q replaced by q^s) using Theorem 3 and Theorem Q. The details are left for the reader. The near vector space $R=K$ gives in these cases a 2-(q^s, k, λ)-design.

Remark. For $s=1$, designs with parameters (e) were derived by Quackenbush [**10**] by a different method, but also using near vector spaces.

Remark. Theorems 2 and 3 are part of my thesis [**8**], and Theorems 4, 5, and 6 generalize Theorems of [**8**].

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Institut für Ang. Mathematik
Universität Freiburg
D-7800 *Freiburg*