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# Distances, Graphs and Designs

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## 1. INTRODUCTION

This paper unifies the treatment of certain problems dealing with intersection matrices of $t$-designs, strongly regular graphs, finite metric spaces, few-distance sets on the Euclidean sphere, and $t$-designs in $Q$-polynomial association schemes.

Central to our theory is the concept of a *distance matrix*. Distance matrices are real, symmetric matrices closely related to finite metric spaces. We classify distance matrices according to two parameters, the *degree* $s$ and the *strength* $t$. The degree is the number of distinct off-diagonal entries, whereas the strength measures the inner regularity of the matrix. A distance matrix without repeated rows which has strength $t$ for all $t \geqslant 0$ is called a *Delsarte matrix*.

Delsarte matrices of degree 2 are essentially the strongly regular graphs, and, in general, a Delsarte matrix is equivalent to a $Q$-polynomial association scheme in the sense of Delsarte [4]. We show that every distance matrix with degree $s$ and strength $t \geqslant 2s-2$ is a multiple of a Delsarte matrix.

For every $t$-design, or transversal $t$-design $\mathscr{B}$ there is an associated distance matrix $C$ of strength $t$ which is closely related to the intersection matrix of $\mathscr{B}$; the degree of $C$ is the number of distinct intersection numbers of $\mathscr{B}$. Using this, we are able to derive results by Majumdar [9] on bounds for the intersection numbers of 2-designs, by Beker and Haemers [1] on 2-designs with intersection number $k-r+\lambda$, and by Cameron [3] on $t$-designs with few intersection numbers.

Finally, spherical $t$-designs introduced by Delsarte, Goethals, and Seidel [5], with the spherical metric, also give rise to distance matrices of strength $t$, which explains the similarity of the theory in [4] and [5].

## 2. DISTANCE MATRICES

Let $X$ be a $w$-set. Since we use $X$ as labelling set for the rows (and columns) of symmetric $w \times w$-matrices we call the elements of $X$ *rows*.

A *distance matrix* (on $X$) is a non-zero real symmetric $w \times w$-matrix $C = (c_{xy})_{x,y \in X}$ with non-negative entries, $c_{xy} \geqslant 0$, and zero diagonal, $c_{xx} = 0$, such that the *distance function* $d(x, y) = c_{xy}^{1/2}$ satisfies the triangular inequality

$$d(x, z) \leqslant d(x, y) + d(y, z), \quad \text{for all } x, y, z \in X. \tag{2.1}$$

If $c_{xy} = 0$ then (2.1) implies $c_{ax} = c_{ay}$ for all $a \in X$, and the converse holds since $c_{xx} = 0$. In particular, $C$ has no repeated rows iff $c_{xy} \neq 0$ for all $x, y \in X$, $x \neq y$. In this case, $d(x, y)$ makes $X$ into a metric space. Conversely, if $X$ is a finite metric space with metric $d(x, y)$ then the matrix $C = (d(x, y)^2)_{x,y \in X}$ is a distance matrix without repeated rows.

We say that two distance matrices $C = (c_{xy})$ on $X$, and $C' = (c'_{xy})$ on $X'$, are *isomorphic* if there are a bijection $\pi : X \to X'$, and a positive number $\gamma$ such that $c_{xy} = \gamma c'_{\pi x, \pi y}$ for all $x, y \in X$. Clearly, isomorphism is an equivalence relation.

We denote the identity of size $m \times m$ by $I_{mm}$, the all-one matrix of size $m \times n$ by $J_{mn}$, and the all-one vector of size $m$ by $\mathbf{j}_m$. If there is no doubt we simply write $I$, $J$, and $\mathbf{j}$. We call any matrix isomorphic to $C \times J_{mm}$ (where $\times$ denotes the Kronecker product) a $m$th *multiple* of



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$C$. A symmetric matrix $C$ is a $m$th multiple of a matrix without repeated rows iff every row of $C$ is repeated exactly $m$ times.

A distance matrix all of whose off-diagonal entries are the same is called *trivial*; all trivial distance matrices are isomorphic to $J-I$.

We say that $x, y \in X$ is an *antipodal pair* of rows of $C$ if $c_{ax}+c_{ay}=c_{xy}$ for all $a \in X$. It is easy to see that the number $\gamma=\frac{1}{2}c_{xy}$ is independent of the antipodal pair, and that if $x, y$ is an antipodal pair then $x, y'$ is an antipodal pair iff $c_{yy'}=0$. In particular, if $C$ has no repeated rows then every row $x$ has at most one *antipodal mate* $x'$ such that $x, x'$ is an antipodal pair.

We say that $C$ is *antipodal* if it has no repeated rows, and every row has an antipodal mate. In this case we may split $X$ into two sets $Y, Z$ such that $Z$ is the set of antipodal mates of $Y$. Then the matrix $D=\gamma J - C|_{Y\times Y}$ has diagonal entries $\gamma$, and no repeated rows, and $C$ can be written as

$$C=\begin{pmatrix}\gamma J-D & \gamma J+D\\ \gamma J+D & \gamma J-D\end{pmatrix}, \tag{2.2}$$

Conversely, every such matrix is antipodal.

If $Y_0$ is a subset of $Y$, and $Z_0=\{x' \mid x\in Y_0\}$ then we can split $X$ into $\bar{Y}=(Y-Y_0)\cup Z_0$, $\bar{Z}=(Z-Z_0)\cup Y_0$; the corresponding matrix $\bar{D}=\gamma J - C|_{\bar{Y}\times\bar{Y}}$ is obtained from $D$ by multiplying the rows and columns corresponding to $Y_0$ by $-1$. This operation is well-known under the name of *switching* (Seidel [13]). Switching with respect to arbitrary subsets is an equivalence relation; and any two switching equivalent matrices $D, \bar{D}$ give rise to isomorphic matrices $C$ via (2.2).

2.1. LEMMA. *Let $C=(c_{xy})$ be a non-zero real symmetric matrix (on $X$) with zero diagonal. If, for some $\gamma>0$, $\gamma J-C$ is positive semi-definite then $C$ is a distance matrix. Moreover, $c_{xy}\leqslant 2\gamma$ for all $x, y\in X$, and $c_{xy}=2\gamma$ implies that $x, y$ is an antipodal pair.*

PROOF. The principal submatrices of dimension 2 and 3 of $\gamma J-C$ are the matrices

$$P_{xy}=\begin{pmatrix}\gamma & \gamma-c_{xy}\\ \gamma-c_{xy} & \gamma\end{pmatrix},\qquad P_{xyz}=\begin{pmatrix}\gamma & \gamma-c_{xy} & \gamma-c_{xz}\\ \gamma-c_{xy} & \gamma & \gamma-c_{yz}\\ \gamma-c_{xz} & \gamma-c_{yz} & \gamma\end{pmatrix}.$$

Hence $0\leqslant \det P_{xy}=2\gamma c_{xy}-c_{xy}^2$, whence $0\leqslant c_{xy}\leqslant 2\gamma$, and $0\leqslant \det P_{xyz}=\gamma(2(c_{xy}c_{xz}+c_{xy}c_{yz}+c_{xz}c_{yz})-c_{xy}^2-c_{xz}^2-c_{yz}^2)-2c_{xy}c_{xz}c_{yz}$, whence $2(c_{xy}c_{xz}+c_{xy}c_{yz}+c_{xz}c_{yz})\geqslant c_{xy}^2+c_{xz}^2+c_{yz}^2$ which can easily be transformed into the triangle inequality for $d(x,y)=c_{xy}^{1/2}$. If $c_{xy}=2\gamma$ then $\det P_{xyz}=-\gamma(c_{xz}+c_{yz}-2\gamma)^2$, which is non-negative only if $c_{xz}+c_{yz}=2\gamma=c_{xy}$. Hence $x, y$ is an antipodal pair.

We now present the examples which relate combinatorial and geometric structures to distance matrices.

A *design* (or *incidence structure*) is a triple $(P, \mathscr{B}, I)$ (loosely written: $\mathscr{B}$) consisting of a set $P$ of $v$ *points*, a set $\mathscr{B}$ of $b$ *blocks*, and a relation $I\subseteq P\times\mathscr{B}$ called *incidence*. We write $p\in B$ or $B\ni p$ if $(p,B)\in I$. Loosely we regard a block $B$ as the set of points incident with $B$. Thus we say $B, B'$ is a pair of *repeated blocks* if $B\neq B'$, and $B$ and $B'$ are incident with exactly the same points. The *block size* is the number of points incident with $B$.

The *incidence matrix* of a design $\mathscr{B}$ is the $v$ $b$-matrix $A$ whose rows are labelled by the points, whose columns are labelled by the blocks, and whose entry in cell $(p, B)$ is $\mathbf{i}_{pB}=1$ or 0 according as $p$ and $B$ are incident or not. The matrix $A^{\mathrm{T}}A$ is called the *intersection matrix* of $\mathscr{B}$, and contains in cell $(B, B')$ the number of points incident with $B$ and $B'$. Hence the diagonal contains just the block sizes of $B$. The off-diagonal elements are called the *intersection numbers* of $\mathscr{B}$. From Lemma 2.1 we directly get the following lemma.

LEMMA 2.2. *Let $A$ be the incidence matrix of a design $\mathscr{B}$ with constant block size $k$. Then $C = kJ - A^{\mathrm{T}}A$ is a distance matrix, or $C = 0$ (We call $C$ the distance matrix of $\mathscr{B}$).*

A *graph* is a pair $(P, \mathscr{G})$ (loosely written: $\mathscr{G}$) consisting of a set $P$ of $v$ *vertices* (or *points*), and a set $\mathscr{G}$ of unordered pairs of points, called *edges*. The *adjacency matrix* of a graph $\mathscr{G}$ is the symmetric matrix $M$ whose rows and columns are labelled by the vertices, and whose entry in cell $(a, b)$ is 1 if $\{a, b\}$ is an edge, and 0 otherwise (some authors use other types of adjacency matrices). The *eigenvalues* of a graph are eigenvalues of its adjacency matrix. Again from Lemma 2.1, we have the following lemma.

LEMMA 2.3. *Let $M$ be the adjacency matrix of a graph $\mathscr{G}$ with smallest eigenvalue $-m$. Then $C = m(J - I) - M$ is a distance matrix, or $C = 0$ (We call $C$ the distance matrix of $\mathscr{G}$).*

In the $d$-dimensional real vector space $\mathbb{R}^d$, we define the standard inner product $\langle x, y\rangle = x_1y_1 + \cdots + x_dy_d$. The *unit sphere* is the set of points $x \in \mathbb{R}^d$ with $\langle x, x\rangle = 1$. If $x, y$ are points on the unit sphere then $\alpha = \arccos\langle x, y\rangle$ is the angle between the vectors $\overrightarrow{0x}$ and $\overrightarrow{0y}$. In particular, $\langle x, y\rangle = 1$ implies $x = y$. If $X$ is a finite set of points then the *Gram matrix* of $X$ is the matrix Gram $X = (\langle x, y\rangle)_{x,y\in X}$. It is well-known that this matrix is positive semi-definite. Hence we obtain the following lemma from Lemma 2.1.

LEMMA 2.4. *Let $X$ be a finite set of $w > 1$ points on the unit sphere in a finite-dimensional real vector space. Then $C = J -$ Gram $(X)$ is a distance matrix without repeated rows (We call $C$ the distance matrix of $X$).*

We note that the metric corresponding to $C$ is (up to a scalar factor) the metric induced on $X$ by the euclidean metric.

Let $X$ be a finite set of points. An (*s-class*) *scheme* on $X$ is a partition of the set $\binom{X}{2}$ of all 2-subsets of $X$ into $s \geq 2$ non-empty classes. Two points $x, y$ are *$\alpha$th associates* if $x \neq y$, and $\{x, y\}$ is in the class with label $\alpha$. We define $k_\alpha(x)$ as the number of $\alpha$th associates of $x$, and $p_{\alpha\beta}(x, y)$ as the number of $z \in X$ which are $\alpha$th associates of $x$, and $\beta$th associates of $y$. If we write $D_\alpha = (d^\alpha_{xy})$ with $d^\alpha_{xy} = 1$ or 0 according as $x$ and $y$ are $\alpha$th associates or not then

$$J - I = \sum_\alpha D_\alpha, \tag{2.3a}$$

$$D_\alpha \mathbf{j} = (k_\alpha(x)), \tag{2.3b}$$

$$D_\alpha D_\beta = (p_{\alpha\beta}(x, y)), \tag{2.3c}$$

$$p_{\alpha\beta}(x, x) = k_\alpha(x)\delta_{\alpha\beta}, \tag{2.3d}$$

where $\delta_{\alpha\beta}$ ($=1$ if $\alpha = \beta$, $=0$ otherwise) is the Kronecker symbol.

A *regular scheme* is a scheme with $k_\alpha(x) = k_\alpha$ for all points $x$, and an *association scheme* is a regular scheme with $p_{\alpha\beta}(x, y) = p^\gamma_{\alpha\beta}$ whenever $x, y$ are $\gamma$th associates.

For later use, we define some invariants of distance matrices.

Let $C$ be a distance matrix on a $w$-set $X$. We call $S = \{c_{xy} | x \neq y\}$ the set of *distance numbers*, and $s = |S|$ the *degree* of $C$. In particular, $C$ is trivial iff $s = 1$.

If $C$ has no repeated rows then $0 \notin S$, and we define the *annihilator polynomial* of $C$ to be the polynomial

$$\mathrm{Ann}_C(x) = w \prod_{\alpha \in S} \left(1 - \frac{x}{\alpha}\right).$$

Then $\mathrm{Ann}_C(0) = w$, and the distance numbers are just the zeros of $\mathrm{Ann}_C(x)$.

The *distribution scheme* of $C$ is the $s$-class association scheme on $X$ defined by calling $x, y$ $\alpha$th associates iff $x \neq y$, and $c_{xy} = \alpha$. The corresponding $(0, 1)$-matrices $D_\alpha$ are called the *distribution matrices* of $C$, and we can express $C$ as

$$C = \sum_{\alpha \in S} \alpha D_\alpha. \tag{2.4}$$

## 3. Strength. Delsarte Matrices

If $A = (a_{xy})$ is a matrix, $i$ is a non-negative integer, and $f$ is a real function, then we define the matrices $A^{(i)} = (a_{xy}^i)$, $f \circ A = (f(a_{xy}))$. Here $0^0 = 1$, so that $A^{(0)} = J$, $A^{(1)} = A$.

Let $t$ be a non-negative integer. We say that a distance matrix $C$ has *strength $t$* if for all non-negative integers $i, j$ with $i + j \leq t$, there is a polynomial $f_{ij}(x)$ of degree $\leq \min(i, j)$ such that

$$C^{(i)}C^{(j)} = f_{ij} \circ C, \; i + j \leq t. \tag{3.1}$$

Isomorphic distance matrices have the same strength. If $C$ has strength $t$ then $C$ has strength $t'$ for all $t' \leq t$. Besides the degree, the maximum strength is the most important characteristic of a distance matrix.

A *Delsarte matrix* is a distance matrix without repeated rows which has strength $t$ for all non-negative integers $t$. It is easy to see that the trivial distance matrices are Delsarte matrices. Examples of non-trivial Delsarte matrices arise, e.g., from certain $t$-designs, strongly regular graphs, and spherical $t$-designs.

Immediately from the definitions, we have the following lemma.

Lemma 3.1.
(i) *Every distance matrix has strength* 0.
(ii) *A distance matrix has constant row sums iff it has strength* 1.

Distance matrices of strength 2 can be characterized by algebraic equations.

Theorem 3.2. *A real symmetric $w \times w$-matrix $C$ with zero diagonal satisfies the equations*

$$CJ = cJ, \qquad C^2 + nC = w^{-1}c(c+n)J, \tag{3.2}$$

*for some positive real numbers $c$, $n$ iff $C$ is a distance matrix of strength* 2. *In this case,*
(i) *$f = c/n$ is an integer,*
(ii) *If $C$ is non-trivial then the matrix $C' = n(J - I) - C$ is also a distance matrix of strength* 2, *with $w' = w$, $c' = n(w-1) - c$, $n' = n$.*

Remark. We call $C'$ the *complement* of $C$. The complement of the complement of $C$ is again $C$.

Proof. If $C$ is a distance matrix of strength 2 then (3.1) for $i = 1, j = 0$, resp. $i = 1, j = 1$ imply the existence of numbers $c$, $a$, $n$ with $CJ = cJ$, $C^2 = aJ - nC$; and multiplication of the second equation by $J$ gives then $a = w^{-1}c(c+n)$. Hence (3.2) holds. Moreover, $c$ is positive since $C$ has only non-negative entries, and $C \neq 0$. Conversely, suppose that $C$ is a real symmetric $w \times w$-matrix with zero diagonal satisfying (3.2) with $c > 0$. Then the matrix $D = w^{-1}cJ - C$ satisfies $D^2 = nD$, hence the only possible eigenvalues of $D$ are $n$ and 0. Suppose that $n$ is an eigenvalue of multiplicity $f$. Then $f$ is a non-negative integer, and $fn = \operatorname{tr} D = c > 0$, whence $f > 0$, $n > 0$, $f = c/n$.

In particular, $D$ is positive semi-definite, and by Lemma 2.1, $C$ is a distance matrix. Now (3.1) for $i \leqslant 1$, $j \leqslant 1$ follows directly from (3.2), and (3.1) for $\{i, j\} = (0, 2\}$, i.e. $C^{(2)}J =$ const. $J$ follows from looking at the diagonal entries of $C^2 = w^{-1}c(c+n)J - nC$. This proves the equivalence, and (i).

Now it is easy to show that $C' = n(J-1) - C$ satisfies equations like (3.2) with the stated parameters.

LEMMA 3.3. *If $C$ is a distance matrix of strength 2 then, with the notation of Theorem 3.2,*

(i) $0 \leqslant c_{xy} \leqslant n$ *for all* $x$, $y$,
(ii) $c_{xy} \in \{0, n\}$ *iff* $c_{xz} = c_{yz}$ *for all* $z \neq x, y$,
(iii) $(2c - n(w-2))/w \leqslant c_{xy} \leqslant 2c/w$ *for all* $x$, $y$ *with* $x \neq y$,
(iv) $c_{xy} = 2c/w$ *iff* $x$, $y$ *is an antipodal pair, and*
(v) $c_{xy} = 2c - n(w-2)/w$ *iff* $x$, $y$ *is an antipodal pair in the complement.*

PROOF. From (3.2), we find

$$\sum_z 1 = w, \tag{3.3a}$$

$$\sum_z c_{xz} = \sum_z c_{yz} = c, \tag{3.3b}$$

$$\sum_z c_{xz}c_{yz} = \frac{c(c+n)}{w} - nc_{xy}, \tag{3.3c}$$

and

$$\sum_z c_{xz}^2 = \sum_z c_{yz}^2 = \frac{c(c+n)}{w}. \tag{3.3d}$$

Hence

$$2c_{xy}^2 \leqslant \sum_z (c_{xz} - c_{yz})^2 = 2c_{xy}n,$$

which gives (i) and (ii). Also, for $x \neq y$,

$$0 \leqslant \sum_{z \neq x,y} \left(c_{xz} + c_{yz} - \frac{2}{w-2}(c - c_{xy})\right)^2$$
$$= \frac{2w}{w-2}\left(c_{xy} - \frac{2c - n(w-2)}{w}\right)\left(\frac{2c}{w} - c_{xy}\right),$$

which gives (iii). Finally,

$$0 \leqslant \sum_z (c_{xz} + c_{yz} - c_{xy})^2 = w\left(c_{xy} - \frac{2c}{w}\right)\left(c_{xy} - \frac{2(c+n)}{w}\right),$$

which gives (iv) since by (iii), $c_{xy} \leqslant 2c/w$. (v) is the complement of (iv).

The next two theorems give sufficient conditions for a distance matrix to be a Delsarte matrix. We also obtain some information on the distribution scheme.

THEOREM 3.4. *Let $C$ be a distance matrix of degree $s$ and strength $t$.*

(i) *If $t \geq s-1$ then the distribution scheme is regular.*

(ii) *If $t \geq s-1$, and $C$ has repeated rows, then $C$ is a multiple of a distance matrix of degree $s-1$ and strength $t$ which has no repeated rows.*

(iii) *If $t \geq 2s-2$ then the distribution scheme is an association scheme, and $C$ is either a Delsarte matrix of degree $s$ or a multiple of a Delsarte matrix of degree $s-1$.*

PROOF. First we remark that $C^{(i)}$ can be expressed by the distribution matrices as

$$C^{(0)} = J,\ C^{(i)} = \sum_{\alpha \in S} \alpha^i D_\alpha + \delta_{i0} I. \tag{3.4}$$

(i) For $i \leq t$, we have $C^{(i)}C^{(0)} = f_{i0} \circ C = \text{const.}J$, whence, for some constant $c_i$, $(\sum_{\alpha \in S} \alpha^i k_\alpha(x))_{x,y} = (C^{(i)} - \delta_{i0} I)C^{(0)} = c_i J$. Hence if $t \geq s-1$ then

$$\sum_{\alpha \in S} \alpha^i k_\alpha(x) = c_i \quad \text{for } i = 0, 1, \ldots, s-1.$$

This system of equations for $k_\alpha(x)$ has a Vandermonde determinant, hence a unique solution; therefore, $k_\alpha(x)$ is independent of $x$, and the distribution scheme is regular.

(ii) If $C$ has repeated rows then 0 is a distance number. (i) implies that $k_0(x) = k_0$ for all $x$; hence every row is repeated exactly $m = k_0 + 1$ times. Therefore, $C$ is a $m$th multiple of another matrix $C_0$ without repeated rows. As a principal submatrix of $C$ this is a distance matrix of degree $s-1$ (since 0 is not a distance number of $C_0$), and from $f \circ (C \times J) = (f \circ C) \times J$, $(C \times J)^{(i)} = C^{(i)} \times J$ follows easily that $C_0$ has strength $t$.

(iii) For $i + j \leq t$, we have $(\sum_{\alpha \in S} \sum_{\beta \in S} \alpha^i \beta^j p_{\alpha\beta}(x, y) = (\sum_{\alpha \in S} \alpha^i D_\alpha)(\sum_{\beta \in S} \beta^j D_\beta) = (C^{(i)} - \delta_{i0} I)(C^{(j)} - \delta_{j0} I) = f_{ij} \circ C - \delta_{i0} C^{(j)} - \delta_{j0} C^{(i)} + \delta_{i0}\delta_{j0} I$. Hence, for $t \geq 2s-2$, there are functions $f_{ij}$ such that for $c_{xy} = \gamma$, $x \neq y$,

$$\sum_{\alpha \in S} \sum_{\beta \in S} \alpha^i \beta^j p_{\alpha\beta}(x, y) = f_{ij}(\gamma) \quad \text{for } i, j = 0, 1, \ldots, s-1.$$

This system of equations for $p_{\alpha\beta}(x, y)$ has a double Vandermonde determinant, hence a unique solution; therefore, for $x \neq y$, $p_{\alpha\beta}(x, y)$ depends only on $\alpha$, $\beta$, and $\gamma = c_{xy}$. Since $s \geq 1$, $t \geq s-1$, so (i) applies, and the distribution scheme is an association scheme.

If $C$ has no repeated rows then the annihilator polynomial $\text{Ann}_C(x)$ is defined, is a polynomial of degree $s$, and satisfies $\text{Ann}_C \circ C = wI$. Hence every $C^{(i)}$, $i \geq s$ can be written as a linear combination of $C^{(0)}, C^{(1)}, \ldots, C^{(s-1)}$, and $I$. Hence, for $i \geq s$, $j \leq s-1$, $C^{(i)}C^{(j)}$ is a linear combination in $C^{(0)}C^{(j)}, \ldots, C^{(s-1)}C^{(j)}$, $C^{(j)}$, hence $= f_{ij} \circ C$ with a polynomial $f_{ij}(x)$ of degree $\leq j$. Repeating this argument with $i$ and $j$ interchanged, we obtain the same conclusion for general $i$, $j$, whence $C$ is a Delsarte matrix of degree $s$.

If $C$ has repeated rows then by (ii), and the above argument, $C$ is a multiple of a Delsarte matrix of degree $s-1$.

THEOREM 3.5. *Let $C$ be a distance matrix, and suppose that $C$ and its complement have strength 3. Then either $C$ is a Delsarte matrix of degree 2, or a multiple of a trivial matrix.*

PROOF. Under the hypothesis, $C^{(2)}C = pC + qJ$ for some real numbers $p$, $q$, and we find

$$\sum_z c_{xz} c_{yz}^2 = p c_{xy} + q. \tag{3.5}$$

Similarly, we obtain from the complement,

$$\sum_z (n - n\delta_{xz} - c_{xz})(n - n\delta_{yz} - c_{yz})^2 = r(n - \delta_{xy} - c_{xy}) + s, \tag{3.6}$$

for appropriate $r$, $s$. If we add (3.5) and (3.6), then straightforward calculations, using Equations (3.3a)–(3.3d), show that, for $x \neq y$, $c_{xy}$ satisfies a quadratic equation with coefficients independent of $x$ and $y$. Hence $c_{xy}$, $x \neq y$, can take at most two values, and $C$ has degree $s \leq 2$. Now the result follows from Theorem 3.4 (iii).

Another consequence of Theorem 3.4 is the following lemma.

LEMMA 3.6. *Let $C$ be a distance matrix of degree $s$, and strength $t \geq \mathrm{Max}\,(2, s-1)$. If $C$ has no repeated rows and contains an antipodal pair then $C$ is antipodal.*

PROOF. By Lemma 3.3 (iv), $\alpha = 2c/w$ is a distance number, and by 3.4 (i), $k_\alpha(x) = k_\alpha > 0$ for all $x$. Hence, again by 3.3 (iv), every point has an antipodal mate, whence $C$ is antipodal.

## 4. THE DISTRIBUTION ALGEBRA

Let $C$ be a distance matrix of degree $s$ without repeated rows, and $S$ be the set of distance numbers of $C$. From now on we shall make use of the conventions

$$S' = S \cup \{0\}, \qquad D_0 = I. \tag{4.1}$$

Using the *Hadamard product* $(a_{ik}) \circ (b_{ik}) = (a_{ik}b_{ik})$ for matrices, we have for the distribution matrices

$$D_\alpha \circ D_\beta = \delta_{\alpha\beta} D_\beta \quad \text{for } \alpha, \beta \in S'. \tag{4.2}$$

Hence the (real) vector space $V$ generated by the $D_\alpha$, $\alpha \in S'$, is an algebra of dimension $s+1$ under the Hadamard product. We call $V$ the *distribution algebra* of $C$. The distribution algebra reflects many properties of $C$. For example, we have the following theorem.

THEOREM 4.1. *The distribution scheme of $C$ is an association scheme iff the distribution algebra is closed under matrix multiplication.*

PROOF. The distribution algebra is closed under matrix multiplication iff $D_\alpha D_\beta = \sum_{\gamma \in S'} p^\gamma_{\alpha\beta} D_\gamma$ for all $\alpha, \beta \in S'$. By (2.3c) and (2.3d), this is equivalent to the fact that the distribution scheme is an association scheme.

We now construct a special basis $E_0, \ldots, E_s$ for the distribution algebra.

LEMMA 4.2. *The distribution algebra $V$ of a distance matrix $C$ contains a canonical chain*

$$V_0 \subset V_1 \subset \cdots \subset V_s = V \tag{4.3}$$

*of vector spaces $V_i$ of dimension $i+1$ defined by*

$$V_i = \{f \circ C \mid f(x) \text{ polynomial of degree} \leq i\} \quad (i = 0, \ldots, s). \tag{4.4}$$

*There are unique matrices $E_0, \ldots, E_s$ satisfying*

$$V_i = \langle C^{(0)}, \ldots, C^{(i)} \rangle = \langle E_0, \ldots, E_s \rangle \quad (i = 0, \ldots, s), \tag{4.5}$$

*and the weak orthogonality relations*

$$\operatorname{tr} E_i E_k = \delta_{ik} \quad (i, k = 0, \ldots, s). \tag{4.6}$$

PROOF. Define $V_i$ by (4.4). Then $V_i = \langle C^{(0)}, \ldots, C^{(i)} \rangle$. Since $C$ has $s+1$ distinct entries, the minimal polynomial $f(x)$ with $f \circ C = 0$ has degree $s+1$; therefore, $C^{(0)}, \ldots, C^{(s)}$ are linearly independent, and $V_i$ has dimension $i+1$. Obviously, $V_0 \subset V_1 \subset \cdots \subset V_s$. By (3.4), $V_s \subseteq V$, and from the dimensions we have $V_s = V$. Hence (4.3) holds.

Now define on $V$ an inner product $(A, B) = \operatorname{tr} AB = \operatorname{tr} A^{\mathrm{T}}B$. This is the canonical inner product on $w \times w$-matrices considered as $w^2$-dimensional vectors, whence it is positive definite. Hence by the Gram–Schmidt algorithm, there is a unique basis $E_0, \ldots, E_s$ of $V$, orthonormal with respect to $(\ ,\ )$, which satisfies (4.5) and (4.6).

REMARK. Since $E_0 \in V_0$, $E_0$ is a multiple of $J$, and (4.6) implies $E_0 = w^{-1}J$. We denote the rank of $E_i$ by $f_i$. The importance of the matrices $E_i$ stems from the following two theorems.

THEOREM 4.3. *$C$ has strength $t$ iff*

$$E_iE_k = \delta_{ik} f_i^{-1/2} E_i \quad \text{for } i, k \leq s,\ i+k \leq t. \tag{4.7}$$

PROOF. (4.7) implies that for $i+k \leq t$, $C^{(i)}C^{(k)} \in V_iV_k = \langle J_{i'}J_{k'} | i' \leq i, k' \leq k \rangle \subseteq V_{\min(i,k)}$, whence $C$ has strength $t$. Conversely, by the above remark, (4.7) is true for $t = 0$. Hence assume by induction that (4.7) holds for $t-1$ instead of $t$. If $i+k = t$ then $E_iE_k \in V_iV_k \subseteq V_{\min(i,k)}$, whence $E_iE_k = \sum_{l=0}^{\min(i,k)} a_{ik}^l E_l$ for certain numbers $a_{ik}^l$. If $i < k$ then for $m \leq i$, $0 = E_iE_kE_m = \sum_{l=0}^{i} a_{ik}^l E_lE_m = a_{ik}^m f_m^{-1/2} E_m$, whence $a_{ik}^m = 0$ for all $m \leq i$, or $E_iE_k = E_kE_i = 0$. If $i = k$ then similarly $a_{ik}^m = 0$ for $m < i$ so that $E_i^2 = a_iE_i$, where $a_i = a_{ii}^i$. Hence the eigenvalues of $E_i$ are 0 or $a_i$, and the multiplicity of $a_i$ equals the rank $f_i$ of $E_i$. Hence $\operatorname{tr} E_i^2 = f_ia_i^2$, and by (4.6), $a_i = f_i^{-1/2}$, which proves (4.7).

REMARK. We have $f_0 = 1$, $E_0 = w^{-1}J$. If $C$ has strength 2 then, in the notation of Theorem 3.2, $f_1 = f = c/n$, $E_1 = n^{-1}f^{1/2}D = (wn)^{-1}f^{1/2}(cJ - wC)$. Thus, 4.3 may be regarded as an extension of Theorem 3.2.

THEOREM 4.4. *Let $C$ be a Delsarte matrix. Then*

(i) *The distribution algebra $V$ is closed under matrix multiplication,*
(ii) *The matrices $E_i' = f_i^{1/2}E_i$ form a basis of mutually orthogonal idempotents of $V$.*
(iii) *There are real $w \times f_i$-matrices $H_i = (h_x^i)$ such that for $i, j = 0, \ldots, s$,*

$$H_iH_i^{\mathrm{T}} = wE_i', \tag{4.8}$$

$$H_i^{\mathrm{T}}H_j = w\delta_{ij}I_{f_if_i}. \tag{4.9}$$

*Moreover, $H = [H_0, \ldots, H_s]$ satisfies $HH^{\mathrm{T}} = H^{\mathrm{T}}H = wI_{ww}$.*

PROOF. (i) and (ii) follow from 4.2 and 4.3. To prove (iii), let $H_i$ be a $w \times f_i$-matrix such that the columns form a set of $f_i$ mutually orthogonal eigenvectors of norm $w^{1/2}$ for the eigenvalue 1 of $E_i'$. By standard linear algebra, (4.8) and (4.9) hold, and the equations for $H$ are an easy consequence of (4.8) and (4.9).

4.2, 4.3, and 4.4 imply that the distribution algebra of a Delsarte matrix is the adjacency algebra of a $Q$-polynomial association scheme in the sense of Delsarte [4]. Conversely, it is easy to see from his definitions that the adjacency algebra of a $Q$-polynomial association scheme contains a distinguished matrix which is a Delsarte matrix. Therefore, Delsarte matrices and $Q$-polynomial association schemes are equivalent concepts. This fact led me to choose Delsarte's name for these matrices since he was the first who considered $Q$-polynomial association schemes. Delsarte also defines $t$-designs in $Q$-polynomial schemes. They correspond to certain distance matrices of strength $t$, as follows.

Let $C$ be a Delsarte matrix of degree $s$. Let $\mathcal{B}$ be a non-empty collection of rows of $C$ (by this we mean a set $\mathcal{B}$ of labels such that each label denotes some row; distinct labels may denote the same row). We define the matrices $C(\mathcal{B})=(c_{xy})_{x,y\in\mathcal{B}}$ and $H_i(\mathcal{B})=(h^i_{xl})_{x\in\mathcal{B},l=1,\ldots,f_i}$. The number of labels in $\mathcal{B}$ will be denoted by $b$. $\mathcal{B}$ is called a *t-design* (of $C$) if $t\leqslant s$, and

$$H_i(\mathcal{B})^{\mathrm{T}}H_k(\mathcal{B})=b\delta_{ik}I_{f_if_i}\quad \text{for } i,k\leqslant s,\ i+k\leqslant t. \tag{4.10}$$

By (4.9), the set of all rows of $C$ is an $s$-design. Also, a $t$-design is an $i$-design for all $i\leqslant t$, and by [4], a $t$-design of a Delsarte matrix is a $t$-design in the corresponding $Q$-polynomial association scheme.

THEOREM 4.5. *Let $\mathcal{B}$ be a t-design of a Delsarte matrix C. Then $C(\mathcal{B})$ is a distance matrix of strength t.*

PROOF. The triangle inequality for $C(\mathcal{B})$ holds since it holds for $C$. By (4.8), $H_iH_i^{\mathrm{T}}=f_i\circ C$ for some polynomial $f_i(x)$ of degree $i$. Hence also $H_i(\mathcal{B})H_i(\mathcal{B})^{\mathrm{T}}=f_i\circ C(\mathcal{B})=J_i(\mathcal{B})$, say. Now (4.10) implies $J_i(\mathcal{B})J_k(\mathcal{B})=b\delta_{ik}J_i(\mathcal{B})$ for $i,k\leqslant s,\ i+k\leqslant t$, and as in the proof of Theorem 4.3 it follows that $C(\mathcal{B})$ has strength $t$.

## 5. COMBINATORIAL EXAMPLES

In this section, we apply the results of Section 3 and 4 to designs, graphs, and spherical designs. Among others, we obtain familiar results by Beker, Bose, Cameron, Delsarte, Majumdar, Goethals and Seidel.

THEOREM 5.1. *Let $C=kJ-A^{\mathrm{T}}A$ be the distance matrix of a design $\mathcal{B}$ with constant block size k and incidence matrix A. Then $JA=kJ$, and*
(i) *C has strength 1 iff $A^{\mathrm{T}}AJ=aJ$ for some constant a;*
(ii) *C has strength 2 iff $AA^{\mathrm{T}}A=nA+\lambda AJ$ for constants $n$, $\lambda$;*
(iii) *C is trivial iff $A^{\mathrm{T}}A=nI+\lambda J$ for constants $n$, $\lambda$.*
In these formulas, $n$ has the same meaning as in 3.2.

PROOF. Obviously, $JA=kJ$.
(i) $C$ has strength 1 iff $CJ=cJ$ for some $c$; so (i) holds with $a=kb-c$.
(ii) $AA^{\mathrm{T}}A=nA+\lambda AJ$ implies (multiply on the left by $J$ resp. $A^{\mathrm{T}}$) $CJ=cJ$, $C^2+nC=c(c+n)/wJ$ with $c=b(k-\lambda)-n$, $w=b$; and $c$, $n>0$ since $C$ is a distance matrix. Conversely, $CJ=cJ$, $C^2=nC=c(c+n)/wJ$ implies $X^{\mathrm{T}}X=0$ for $X=AA^{\mathrm{T}}A-nA-\lambda AJ$, where $\lambda=k-(c+n)/b$. Hence $X=0$.
(iii) $C$ is trivial iff $C=n(J-I)$ for some constant $n$; so (iii) holds with $\lambda=k-n$.

The *dual* of a design is obtained by interchanging the roles of points and blocks, and reversing incidence. The dual of a design with constant block size satisfying the conditions of (i), (ii), or (iii) or Theorem 5.1 is called a *weak 1-design*, *weak $1\frac{1}{2}$-design*, or *weak 2-design*, respectively. A weak 2-design is the same as an $(r,\lambda)$-design, and for weak $1\frac{1}{2}$-designs see [11].

A *$t$-$(v,k,\lambda)$-design* is a design on $v$ points with constant block size $k$ such that any $t$ distinct points are in exactly $\lambda$ blocks. A $t$-design is also an $i$-design for all $i\leqslant t$. A *transversal $t$-$(v,k,\lambda)$-design* is a design on $v$ points, the points being partitioned into $k$ classes of $v/k$ points each, such that each block contains exactly one point from every class (i.e. it is a *transversal* of the partition), and any $t$ points from distinct classes are in exactly $\lambda$ blocks. A transversal $t$-design is also a transversal $i$-design for all $i\leqslant t$. A *$1\frac{1}{2}$-design* (or *partial geometric design*) is a design whose incidence matrix $A$ satisfies $AJ=rJ$, $JA=kJ$, $AA^{\mathrm{T}}A=nA+\alpha J$ for certain integers $r$, $k$, $n$, $\alpha$.

Every transversal 1-design is a 1-design. 2-designs, transversal 2-designs, and the duals of $1\frac{1}{2}$-designs are $1\frac{1}{2}$-designs. Moreover, every 1-design ($1\frac{1}{2}$-design, 2-design) is a weak 1-design ($1\frac{1}{2}$-design, 2-design). Much is known about $t$-designs and $1\frac{1}{2}$-designs; for a summary, see, e.g., Neumaier [10].

The *complete designs* $J(k, v)$ have $v$ points, and all $k$-sets of points as blocks. The *complete transversal designs* $H(k, q)$ have $v = kq$ points, partitioned into $k$ classes of $q$ points each, and all transversals of the partition as blocks. The distribution schemes of $J(k, v)$ and $H(k, q)$ are usually called the *Johnson* resp. *Hamming schemes*.

THEOREM 5.2. (Delsarte [4])

(i) *The distance matrix $C_J$ of a complete design $J(k, v)$ is a Delsarte matrix of degree $s = \min(k, v-k)$. For $v \geq 2k$, a collection $\mathcal{B}$ of rows of $C_J$ is a t-design of $C_J$ iff $\mathcal{B}$ is a $t$-$(v, k, \lambda)$-design, for some integer $\lambda$.*

(ii) *The distance matrix $C_H$ of a complete transversal design $H(k, q)$ is a Delsarte matrix of degree k. A collection $\mathcal{B}$ of rows of $C_H$ is a t-design of $C_H$ iff $\mathcal{B}$ is a transversal $t$-$(qk, k, \lambda)$-design, for some integer $\lambda$.*

THEOREM 5.3. *The distance matrix of a t-design or a transversal t-design has strength t.*

PROOF. By 5.2 and 4.5.

COROLLARY 5.4. (Cameron [3]). *The blocks of a $(2s-2)$-design with s intersection numbers carry a natural association scheme.*

COROLLARY 5.5. *The blocks of a transversal $(2s-2)$-design with s intersection numbers carry a natural association scheme.*

PROOFS. By 5.3 and 3.4 (iii).

Note that by Theorem 5.3, the distance matrix of a $1\frac{1}{2}$-design has strength 2; and since there are many $1\frac{1}{2}$-designs which are neither 2-designs nor transversal 2-designs, the converse of Theorem 5.3 is not valid.

COROLLARY 5.6. (Majumdar [9], Beker and Haemers [1]). *Two blocks of a 2-$(v, k, \lambda)$-design $\mathcal{B}$ intersect in at least $k-r+\lambda$ points, where $r = \lambda(v-1)/k-1$. Moreover, the relation $\equiv$ on the blocks defined by $A \equiv B$ iff $A = B$ or $|A \cap B| = k-r+\lambda$ is an equivalence relation.*

PROOF. By 5.1, the distance matrix $C'$ of $\mathcal{B}$ has strength 2, with $n' = r-\lambda$, $\lambda' = k/r$. Therefore, its complement $C = n'(J-I)-C'$ is a distance matrix with off-diagonal entries $c_{AB} = n'-c_{AB'} = r-\lambda-k+|A \cap B|$. Now $c_{AB} \geq 0$ implies $|A \cap B| \geq k-r+\lambda$, and $|A \cap B| = k+r+\lambda$ holds iff $c_{AB} = 0$, i.e. iff $A, B$ are repeated rows of $C$. But repeatedness is an equivalence relation.

For graphs, we can give a theorem similar to 5.1.

THEOREM 5.7. *Let $C = m(J-I)-M$ be the distance matrix of a graph $\mathcal{G}$ with smallest eigenvalue $-m$, and adjacency matrix $M$. Then*

(i) *$C$ has strength 1 iff $MJ = kJ$ for some $k$;*

(ii) *$C$ has strength 2 iff $M^2 = (\lambda-\mu)M+(k-\mu)I+\mu J$ for some $\lambda$, $\mu$, $k$;*

(iii) *$C$ is a multiple of a trivial distance matrix iff $\mathcal{G}$ is the disjoint union of complete graphs of the same size.*

PROOF. Similar to the proof of 5.1. Observe that the diagonal of $M^2$ contains the entries of $MJ$ since $M$ is a (0, 1)-matrix.

A graph $\mathscr{G}$ is called *regular*, resp. *strongly regular* if the condition of (i), resp. (ii) of Theorem 5.7 is satisfied. (i) means that every vertex is adjacent to $k$ other vertices, and (ii) means that in addition, the number of vertices adjacent to two distinct vertices $a$ and $b$ is $\lambda$ or $\mu$ according as $a$ and $b$ are adjacent or not. A lot of results about strongly regular graphs is contained in Hubaut [7], Neumaier [12], and Seidel [13].

Complementary strongly regular graphs give rise to complementary distance matrices.

If $\mathscr{G}$ is a graph on $X$ then there is a canonical scheme of degree 2 on $X$ which has adjacent points as first associates, and non-adjacent points as second associates. This scheme is essentially the distribution scheme of the corresponding distance matrix, and is regular, resp. an association scheme iff the graph is regular, resp. strongly regular. Of course, every scheme of degree 2 comes from a graph in this way.

THEOREM 5.8. *The distance matrix of a strongly regular graph which is connected and not complete is a Delsarte matrix of degree 2. Conversely, every Delsarte matrix of degree 2 is equivalent to the distance matrix of a strongly regular graph which is connected and not complete.*

PROOF. The distance matrix $C$ of a strongly regular graph $\mathscr{G}$ has strength 2 and degree 2. If $C$ is not a Delsarte matrix then by Theorem 3.4 (iii), $C$ is a multiple of a trivial distance matrix, whence, by 5.7 (iii), $\mathscr{G}$ is complete or disconnected.

Conversely, if $C$ is a Delsarte matrix of degree 2 with distance numbers $\alpha_1 < \alpha_2$ then the graph $\mathscr{G}$ whose vertices are the rows of $C$, and whose edges are the pairs $(x, y)$ with $c_{xy} = \alpha_1$ has adjacency matrix $M = (\alpha_2 - \alpha_1)^{-1}(\alpha_2(J - I) - C)$, hence smallest eigenvalue $-m = -(\alpha_2 - \alpha_1)^{-1}\alpha_2$. Therefore, $M$ has distance matrix $(\alpha_2 - \alpha_1)^{-1}C$, and by 5.7 (ii)–(iii), $\mathscr{G}$ is strongly regular, connected, and not complete.

In many cases, interesting graphs can be found from designs. The *block graph* of a $1\frac{1}{2}$-design $\mathscr{B}$ with two intersection numbers $\mu_1 > \mu_2$ is the graph whose vertices are the blocks of $\mathscr{B}$, adjacent iff they intersect in $\mu_1$ points. A $1\frac{1}{2}$-design with intersection numbers 0 and 1 is a *partial geometry* [2], and in this case blocks are called *lines*, and the block graph is called the *line graph*. A 2-design with two intersection numbers is called *quasi-symmetric* [6]. From Theorem 5.1 (ii) and Theorem 5.7 (ii) we now obtain without difficulty the following well-known results

COROLLARY 5.9. ([2], [3], [6], [10]). *The block graph of a $1\frac{1}{2}$-design with two intersection numbers is strongly regular. This holds in particular for the block graph of a quasi-symmetric 2-design, for the line graph of a partial geometry, and for the line graph of a 2-$(v, k, 1)$-design.*

If a 2-design $\mathscr{B}$ has three intersection numbers $\mu_1$, $\mu_2$, and $\mu_3 = k - r + \lambda$ then we may form the equivalence classes of $\equiv$ of 5.6. By 3.4(i), these block classes have the same size, so the complement of the distance matrix of $\mathscr{B}$ is a multiple of a distance matrix of degree 2 and strength 2. Hence we have a strongly regular graph on the classes of $\mathscr{B}$, called the *class graph* of $\mathscr{B}$.So we have the following corollary.

COROLLARY 5.10. (Beker and Haemers [1]). *The class graph of a 2-design with three intersection numbers, one of which is $k - r + \lambda$ is strongly regular, and all classes have the same size.*

The results analogous to 5.9, 5.6, and 5.10 for transversal designs are summarized in the following corollary.

COROLLARY 5.11.
(i) *The block graph of a transversal* 2*-design with two intersection numbers is strongly regular.*
(ii) *Two blocks of a transversal* $2$-$(v, k, \lambda)$*-design* $\mathcal{B}$ *intersect in at least* $k-r$ *points, where* $r = k^{-1}v\lambda$. *If* $\mathcal{B}$ *has three intersection numbers, one of which is* $k-r$ *then the blocks of* $\mathcal{B}$ *can be partitioned into classes of the same size such that the class graph of* $\mathcal{B}$ *is strongly regular.*

PROOF. Similar to the proofs of 5.9, 5.6 and 5.10.

Finally we mention some results on the spherical case. The details follow easily from Delsarte, Goethals, and Seidel [5].

THEOREM 5.12.
(i) *The distance matrix of a spherical set* $X$ *of points has strength* 1 *iff the centre of mass of* $X$ *is in the origin.*
(ii) *The distance matrix of a spherical* $t$*-design has strength* $t$.
(iii) *Every distance matrix of strength* 2 *without repeated rows is isomorphic to the distance matrix of a spherical* 2*-design.*

Again, many of the results of [5] can now be obtained as corollaries from 5.12 and the above results.

Note that there is no analogue of (iii) for the case $t>2$. Distance matrices of strength $t>2$ yield, in general, only spherical 2-designs. It is not known what happens in case $t=1$.

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