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Strongly regular graphs with smallest eigenvalue $-m$

By
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1. Definitions and well-known results. All our graphs are undirected, without loops and multiple edges. A graph Γ is *strongly regular* (or a SRG) if

(i) every vertex is adjacent to k vertices,

(ii) the number of vertices adjacent to any two adjacent vertices is λ,

(iii) the number of vertices adjacent to any two nonadjacent vertices is μ.

We denote the number of vertices by v, and assume always the nondegeneracy condition $2 \leqq k \leqq v - 3$. The vertex set of Γ is denoted by P. A counting argument gives

(1) $$\mu(v-1-k) = k(k-1-\lambda).$$

We call two vertices *first* (*second*) *associates* if they are distinct and adjacent (nonadjacent). Then the number of vertices z which are i-th associates to x and j-th associates to y is p_{ij}^k if x and y are k-th associates; here

$$p^1 = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{bmatrix} = \begin{bmatrix} \lambda & k-1-\lambda \\ k-1-\lambda & \lambda+v-2k \end{bmatrix},$$

$$p^2 = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix} = \begin{bmatrix} \mu & k-\mu \\ k-\mu & \mu+v-2k-2 \end{bmatrix}.$$

The *complementary graph* Γ' with the same points, adjacent if they are distinct and nonadjacent in Γ is also strongly regular, with parameters

$$\bar{v} = v, \quad \bar{k} = v-1-k, \quad \bar{\lambda} = \mu+v-2k-2, \quad \bar{\mu} = \lambda+v-2k.$$

The *adjacency matrix* $M = (m_{ab})_{a,b\in P}$ of Γ has $m_{ab} = 1$ if ab is an edge, $=0$ otherwise. M satisfies the equations

(2) $$M^2 = (\lambda-\mu)M + (k-\mu)I + \mu J, \quad MJ = kJ,$$

and has the eigenvalues k, $n - m$, $-m$ with multiplicities 1, f, $v - 1 - f$, where

(3) $$n^2 = (\mu-\lambda)^2 + 4(k-\mu), \quad n > 0, \quad m = \tfrac{1}{2}(n+\mu-\lambda),$$

$$f = \tfrac{1}{2}\left(v-1-\frac{1}{n}(2k-(v-1)(\mu-\lambda))\right).$$

*) Part of this work was done while the author was at Westfield College, London.

1.1. Lemma. *The parameters of a* SRG *can be expressed as*

$$(4)\qquad \begin{aligned} v &= \mu + \overline{\mu} + n(2m-1) - 2m(m-1),\\ f &= \frac{1}{n}\left((m-1)\mu + m\overline{\mu} + m(m-1)(2n-2m+1)\right),\\ k &= \mu + m(n-m), \quad \bar{k} = \overline{\mu} + (m-1)(n+1-m),\\ \lambda &= \mu + n - 2m, \quad \bar{\lambda} = \overline{\mu} + 2m - n - 2,\\ P^1 &= \begin{bmatrix} \mu + n - 2m & (m-1)(n+1-m)\\ (m-1)(n+1-m) & \overline{\mu} \end{bmatrix},\\ P^2 &= \begin{bmatrix} \mu & m(n-m)\\ m(n-m) & \overline{\mu} + 2m - n - 2 \end{bmatrix}, \end{aligned}$$

where m, n, μ, $\overline{\mu}$ *satisfy the restrictions*

$$(5)\qquad \begin{aligned} &\mu\overline{\mu} = m(m-1)(n+1-m)(n-m),\\ &1 \leqq m \leqq n,\\ &\mu \geqq \max(0, 2m-n),\\ &\overline{\mu} \geqq \max(0, n-2m+2). \end{aligned}$$

Moreover, if $\mu \neq 0$ *then*

$$(6)\qquad \begin{aligned} v &= \frac{1}{\mu}\left(\mu + (m-1)(n-m)\right)\left(\mu + m(n+1-m)\right),\\ f &= \frac{m-1}{\mu n}\left(\mu + m(n-m)\right)\left(\mu + m(n+1-m)\right),\\ \bar{k} &= \frac{(m-1)(n+1-m)}{\mu}\left(\mu + m(n-m)\right). \end{aligned}$$

also $n^2 = \frac{vkl}{fg}$

Proof. The expressions for the parameters, and the equation for $\mu\overline{\mu}$ can be easily verified using (1) and (2), $1 \leqq m \leqq n$ follows from the fact that p_{12}^1 and p_{12}^2 are nonnegative, and the inequalities for μ and $\overline{\mu}$ come from the fact that the other p_{ij}^k are nonnegative.

A *conference graph* (or *pseudo-cyclic* graph) is a SRG with $v = 4\mu + 1$, $k = 2\mu$, $\lambda = \mu - 1$, and hence $n = \sqrt{4\mu + 1}$, $m = \frac{1}{2}(1 + \sqrt{4\mu + 1})$. It is not difficult to prove from (1) and (2):

1.2. Lemma. *The parameters* m, n *of a* SRG *which is not a conference graph are integers.*

A straightforward proof yields also

1.3. Lemma.

(i) $m = 1$ *iff* $\mu = 0$ *iff* Γ *is disconnected iff* Γ *is the union of at least 2 pairwise disjoint* n*-cliques.*

(ii) $m = n$ *iff* $\overline{\mu} = 0$ *iff* Γ *is a complete multipartite graph with classes of size* m.

1.4. Lemma. *The complement of a* SRG *with parameters* m, n, μ *has parameters* $m' = n + 1 - m$, $n' = n$, $\mu' = \overline{\mu}$.

2. Krein condition and absolute bound. If Γ is a SRG such that Γ and its complement are connected (i.e. $1 < m < n$) then there is another notation for the parameters. Hubaut [8] and Seidel [12] use

their parameters for the complement	my parameters
n	v
r	$n - m$
s	$-m$
f	f
g	$v - 1 - f$

Hubaut's paper contains constructions for most of the known SRGs, whereas Seidel's paper is a survey of theoretical results on SRGs. In particular, Seidel reproves two necessary conditions for the parameters of a SRG which he calls the Krein condition and the absolute bound.

2.1. Lemma (*Krein condition*). *If* $1 < m < n$ *then*

$$\mu(n - m(m-1)) \leqq (m-1)(n-m)(n + m(m-1)). \tag{7}$$

2.2. Lemma (*Absolute bound*). *If* $1 < m < n$ *then*

$$v \leqq \tfrac{1}{2} f(f+3). \tag{8}$$

Equality in this conditions imply certain extra geometrical properties.

3. The bound for μ.

3.1. Theorem. *For a nontrivial SRG with integral smallest eigenvalue* $-m$, $1 < m < n$, *we have*

$$\mu \leqq m^3(2m - 3). \tag{9}$$

Equality implies $n = m(m-1)(2m-1)$.

Remark. We call (9) the *μ-bound.*

Proof. Fix m, $1 < m < n$. Then $\mu\overline{\mu} \neq 0$.

(i) If $n \leqq 2m - 1$ then $\overline{\mu} \geqq 1 \rightarrow \mu \leqq m(m-1)(n+1-m)(n-m) \leqq m^2(m-1)^2 < m^3(2m-3)$.

(ii) If $2m - 1 \leqq n \leqq m^2 + m - 2$ then

$$\overline{\mu} \geqq n - 2m + 2 \rightarrow \mu \leqq \frac{m(m-1)(n+1-m)(n-m)}{n - 2m + 2}.$$

This expression is convex in n, hence has its maximum at either $n = 2m - 1$, or $n = m^2 + m - 2$. Therefore,

$$\mu \leqq \max(m^2(m-1)^2, \quad (m^2-2)(m^2-1)) < m^3(2m-3).$$

(iii) If $m^2 + m - 2 \leqq n \leqq m(m-1)(2m-1)$ then the Krein condition implies $\mu \leqq \dfrac{(m-1)(n-m)(n+m(m-1))}{n-m(m-1)}$. This expression is again convex in n, whence $\mu \leqq \max((m^2-2)(m^2-1),\, m^3(2m-3)) = m^3(2m-3)$, and equality holds iff $n = m(m-1)(2m-1)$.

(iv) If $n > m(m-1)(2m-1)$ then $\mu < m(n-m)$; for $\mu \geqq m(n-m)$ gives a contradiction with the Krein condition. We show that the assumption $\mu \geqq m^3(2m-3)$ conflicts with the absolute bound. In fact, using (6), we find

$$\frac{2v}{f} = \frac{2n}{m-1} \cdot \frac{\mu + (m-1)(n-m)}{\mu + m(n-m)} = 2\mu \cdot \frac{n-m}{\mu m} + 2 + \frac{\mu + m^2}{m^2(m-1)} + $$
$$+ \frac{(\mu - m^2)(m(n-m)-\mu)}{m^2(m-1)(\mu + m(n-m))} > 2\mu \cdot \frac{n-m}{\mu m} + 2 + \frac{\mu+m^2}{m^2(m-1)},$$

and, in a similar way,

$$f < m^3(m-1) \cdot \frac{n-m}{\mu m} + 3m(m-1) - \frac{2m^2(m-1)(2m-1)}{\mu + m^2}.$$

Inserting this into the absolute bound $\dfrac{2v}{f} \leqq f + 3$ gives

$$(2\mu - m^3(m-1)) \frac{n-m}{\mu m} < 3m^2 - 3m + 1 -$$
$$- \frac{\mu + m^2}{m^2(m-1)} - \frac{2m^2(m-1)(2m-1)}{\mu + m^2}.$$

The right hand side is concave in μ, with a maximum at
$\mu = -m^2 + m^2(m-1)\sqrt{4m-2} \leqq m^3(2m-3)$,
hence (under our assumption $\mu \geqq m^3(2m-3)$) it is $\leqq m^2(3m-5)$. The left hand side is $> \dfrac{2\mu - m^3(m-1)}{m^2} \geqq m^2(3m-5)$, contradiction.

Remarks. 1. Hoffman [7] proved a bound $\mu \leqq f(m)$ for $1 < m < n$, with very large $f(m)$.

2. For $m = 2, 3$, there are SRGs which satisfy (9) with equality.

4. Partial geometries. A *geometric* 1-*design* consists of a set P of *points*, a set L of *lines*, and an incidence relation I between points and lines such that

(i) Every point is incident with (or *on*) exactly $R \geqq 2$ lines,
(ii) Every line is incident with (or *contains*) exactly $K \geqq 2$ points,
(iii) Two distinct points are incident with at most one line.

The *dual* of a geometric 1-design is obtained by interchanging the role of points and lines, and is again a geometric 1-design (with R and K interchanged). Two points (lines) are called *adjacent* if they are on a common line (contain a common point). This defines a graph on the set of points (lines), and we call it the *point graph* (*line graph*) of the design. Point graph and line graph are dual concepts.

A 2-$(v, K, 1)$-*design* is a geometric 1-design on v points with the property that any two distinct points are on a unique line. Then $R = \frac{v-1}{K-1}$, and the number of lines is $b = \frac{v(v-1)}{K(K-1)}$. The line graph of a 2-$(m + n(m-1), m, 1)$-design with $n \geqq m + 1$ is called a *Steiner graph* $S_m(n)$. An $S_2(n)$ is also called a *triangular graph* $T(n+2)$. By famous theorems of Wilson [15] and Hanani [5], $2-(v, K, 1)$-designs exist for all $v \geqq v_0(K)$ with $K-1 \mid v-1$, $K(K-1) \mid v(v-1)$. We have $v_0(K) = K^2 - K + 1$ for $K \leqq 5$. Hence

4.1. Lemma. *Steiner graphs* $S_m(n)$ *exist for all* $n \geqq n_1(m)$ *with* $m \mid n(n+1)$, *and* $n_1(m) = m + 1$ *for* $m \leqq 5$.

A $2(v, K, 1)$-transversal design is a geometric 1-design on v points which can be partitioned into K classes of $R = v/K$ points each such that two distinct points are on a line iff they are in distinct classes. The line graph of a 2-$(mn, m, 1)$-transversal design with $n \geqq m + 1$ is called a *Latin square graph* $LS_m(n)$ since a $2-(mn, m, 1)$-transversal design is equivalent to $m-2$ mutually orthogonal Latin squares of order n. An $LS_2(n)$ is also called a *lattice graph* $L_2(n)$. By a famous theorem of Chowla, Erdös, and Straus [4] 2-$(m\,n, m, 1)$- transversal designs exist for $n > n_2(m)$, and $n_2(m) = m - 1$ for $m \leqq 3$. Hence we have

4.2. Lemma. *Latin square graphs* $LS_m(n)$ *exist for all* $n > n_2(m)$, *and* $n_2(m) = m + 1$ *for* $m \leqq 3$.

A *partial geometry* $PG(R, K, \alpha)$ is a geometric 1-design with the property that for any nonincident point-line pair (p, l), there are exactly $\alpha \geqq 1$ points on l adjacent to p. The dual of a $PG(R, K, \alpha)$ is a partial geometry $PG(K, R, \alpha)$. A partial geometry with $\alpha = 1$ is called a *generalized quadrangle*. The following lemmas are well-known and straightforward.

4.3. Lemma. *For a partial geometry* $PG(R, K, \alpha)$,

$$\alpha \leqq \min(R, K). \tag{10}$$

$\alpha = K$ iff the partial geometry is a 2-$(R(K-1)+1, K, 1)$-design, and $\alpha = K - 1$ iff the partial geometry is a 2-$(RK, K, 1)$-transversal design.

4.4. Theorem (*Bose* [1]). *The point graph of a partial geometry* $PG(R, K, \alpha)$ *with* $\alpha < K$ *is a* SRG *with parameters*

$$m = R, \quad n = R + K - \alpha - 1, \quad \mu = \alpha R. \tag{11}$$

In particular, a Steiner graph $S_m(n)$ is strongly regular with parameters $m, n, \mu = m^2$, and a Latin square graph $LS_m(n)$ is strongly regular with parameters $m, n, \mu = m(m-1)$.

By (10) and 4.4, the point graph of a partial geometry has $m \mid \mu \leqq m^2$. Any SRG with $m \mid \mu \leqq m^2$ is called *pseudo geometric*, and *geometric* if it is the point graph of a partial geometry. A SRG is called a *pseudo Steiner graph* $S'_m(n)$ if $\mu = m^2$, and a *pseudo Latin square graph* $LS'_m(n)$ if $\mu = m(m-1)$.

The known conditions for SRGs allow to generalize an inequality of Higman [6] for generalized quadrangles.

4.5. Theorem. *For a partial geometry $PG(R, K, \alpha)$ with $\alpha < K - 1$,*

$$R - 1 \leqq (K-\alpha)^2(2\alpha - 1), \tag{12}$$

and equality implies $\alpha = 1$ or $K = 2\alpha + 1$.

Proof. The complement of the point graph has parameters

$$m = K - \alpha, \quad n = R + K - \alpha - 1,$$

$$\mu = \frac{1}{\alpha}(R-1)(K-\alpha)(K-\alpha-1).$$

For $K \geqq 2\alpha + 1$, we employ the Krein condition which yields

$$(K - 2\alpha)(R-1) \leqq (K-\alpha)^2(K-2). \tag{13}$$

Since $K - 2 = (2\alpha - 1)(K - 2\alpha) - 2(\alpha - 1)(K - 2\alpha - 1) \leqq (2\alpha - 1)(K - 2\alpha)$, (12) follows. For $K \leqq 2\alpha$, we use theorem 3.1 which gives

$$(K - \alpha - 1)(R - 1) \leqq (K-\alpha)^2 \alpha (2K - 2\alpha - 3). \tag{14}$$

Since $\alpha(2K - 2\alpha - 3) = (2\alpha - 1)(K - \alpha - 1) - (2\alpha + 1 - K) < (2\alpha - 1)(K - 2\alpha)$, (12) follows again.

Using properties of m-claws, Neumaier [9] proves

4.6. Theorem. *A* SRG *with smallest eigenvalue $-m$, $m > 1$ integral, is geometric if*

$$n > \max(2(m-1)(\mu + 1 - m), \tfrac{1}{2}m(m-1)(\mu+1) + m - 1). \tag{15}$$

This result has been proved by Bose [1] under the additional assumption that the graph is pseudo geometric (which is essential for his proof). It turns out that inequality (15) can be satisfied only in case of pseudo Latin square graphs and pseudo Steiner graphs:

4.7. Theorem. *Let Γ be a* SRG *with smallest eigenvalue $-m$, $m > 1$ integral.*

(i) *If $\mu = m(m-1)$, i.e. Γ is a $LS'_m(n)$, and $n > \frac{1}{2}(m-1)(m^3 - m^2 + m + 2)$ then Γ is a Latin square graph $LS_m(n)$ (Bruck [2]).*

(ii) *If $\mu = m^2$, i.e. Γ is a $S'_m(n)$, and $n > \frac{1}{2}(m-1)(m^3 + m + 2)$ then Γ is a Steiner graph $S_m(n)$* (Bose [1]).

(iii) *If $\mu \neq m(m-1)$, m^2 then*

$$n \leqq \tfrac{1}{2}m(m-1)(\mu+1) + m - 1. \tag{16}$$

Remark. We call condition (iii) the *claw bound.*

Proof. Suppose first that (15) holds. Then theorem 4.6 implies that there is a partial geometry with $R = m$, $K = \frac{\mu}{m} + n + 1 - m$, $\alpha = \frac{\mu}{m}$. Now

$$K - 1 > 2(R - \alpha) R(\alpha - 1) - (R - \alpha) \tag{17}$$

since otherwise

$$n = R + K - \alpha - 1 \leqq 2(m - \alpha) m(\alpha - 1) \leqq$$
$$\leqq 2(m-1)(m(\alpha - 1) + 1) = 2(m-1)(\mu + 1 - m).$$

If $\mu \neq m, m(m-1), m^2$ then $2 \leqq \alpha < R - 1$ since α is an integer. For $R \leqq 2\alpha$, (17) contradicts the dual of (12), and for $R \geqq 2\alpha + 1$, (17) contradicts the dual of (13). If $\mu = m$ then the dual of (12) says $K - 1 \leqq (R-1)^2$, i.e. $n \leqq m(m-1)$, contradicting (15). Hence $\mu = m(m-1)$, or $\mu = m^2$, and we get (i) and (ii).

Now suppose that (15) and (16) fail. Then

$$\tfrac{1}{2} m(m-1)(\mu + 1) + m - 1 < n \leqq 2(m-1)(\mu + 1 - m), \tag{18}$$

whence $0 < 2(m-1)(2m+1) < (4-m)(m-1)(\mu+1)$, or $m < 4$. For $m = 2$ and $m = 3$, a somewhat tedious calculation shows that (18) contradicts the absolute bound (this part of the argument is due to Brouwer [16]). Hence (16) holds.

5. The characterization theorem. In this section we give a new proof of the following theorem by Sims (cf. Ray-Chaudhuri [10]).

5.1. Theorem (*Sims*). *The* SRG*s with smallest eigenvalue* $-m$, $m \geqq 2$ *integral, are the following*:

(a) *Complete multipartite graphs with s classes of size m,*

(b) *Latin square graphs* $LS_m(n)$,

(c) *Steiner graphs* $S_m(n)$,

(d) *Finitely many other graphs.*

Proof. Suppose first that $n > m^4(m-1)^2$. Since, by 3.1, $\mu \leqq m^3(2m-3)$, we have $n > \frac{m(m-1)}{2}(\mu + 1) + m - 1$. Hence the claw bound implies that $\mu = m(m-1)$ or $\mu = m^2$. But then theorem 4.7 implies that we have case (b) or (c). Suppose now that $n \leqq m$. Then $n = m$, and by 1.3 (b), we have case (a). Finally suppose that $m < n < m^5(m-1)$. By 3.1, $1 \leqq \mu \leqq m^3(2m-3)$ whence there are only finitely many possible triples (m, n, μ) with the given m. For each of them there are only finitely many graphs, so that (d) holds.

5.2. Theorem. *The* SRG*s not covered in theorem* 5.1 *are the following*:

(e) *Conference graphs,*

(f) *The union of s pairwise disjoint n-cliques.*

Proof. By 1.2 and 1.3.

We look more closely at the six possibilities:

a) These graphs have $n = m$, $\mu = m(s-1)$, and exist for all $s \geqq 2$.

b) These graphs have $\mu = m(m-1)$, and exist for all sufficiently large n, by Chowla, Erdös, and Straus [4].

c) These graphs have $\mu = m^2$. By 1.1 (5), they can exist only for $m \mid n(n+1)$. By Wilson [15], they exist for all sufficiently large n with $m \mid n(n+1)$.

(d) These graphs fall into three classes:

(d_1) Pseudo Latin square graphs $LS'_m(n)$ which are not Latin square graphs. They have $\mu = m(m-1)$, and by theorem 4.7, $n \leqq \frac{1}{2}(m-1)(m^3 - m^2 + m + 2)$.

(d_2) Pseudo Steiner graphs $S'_m(n)$ which are not Steiner graphs. They have $\mu = m^2$, and by theorem 4.7, and 1.1 (5), $m \mid n(n+1)$, $n \leqq \frac{1}{2}(m-1)(m^3 + m + 2)$.

(d_3) Graphs with exceptional parameter sets, see below.

(e) These graphs have $v = 4\mu + 1$, $k = 2\mu$, $\lambda = \mu - 1$, and exist for infinitely many values of μ. v has to be a sum of two integral squares (see e.g. [8]).

(f) These graphs have $m = 1$, $\mu = 0$, and exist for all positive integers s and n.

A *parameter set* is a 10-tuple $\Pi = (m, n, \mu, \bar{\mu}, f, v, k, \bar{k}, \lambda, \bar{\lambda})$ of integers satisfying the relations (4) and (5) of section 1. A parameter set Π is *admissible* if it satisfies the Krein condition (lemma 2.1), the absolute bound (lemma 2.2), the μ-bound (theorem 3.1), and the claw bound (theorem 4.7 (iii)). An admissible parameter set Π with $1 < m < n$, $\mu \neq m^2$, $m(m-1)$ is called *exceptional*. By theorem 5.1 (d), there are only finitely many exceptional parameter sets for each integer $m \geqq 2$.

In table 1 we state the three exceptional parameter sets with $m = 2$.

Table 1. Exceptional parameter sets with $m = 2$.

No.	m	n	μ	$\bar{\mu}$	f	v	k	$\bar{k}$	λ	$\bar{\lambda}$	Examples
1	2	3	1	4	5	10	3	6	0	3	Peterson graph
2	2	4	6	2	5	16	10	5	6	0	Clebsch graph
3	2	6	8	5	6	27	16	10	10	1	Schläfli graph

The uniqueness of the three exceptional graphs has been shown by Seidel [11]. Shrikhande [13] showed that there is a unique graph with $m = 2$ in class (d_1), namely a $LS'_2(4) = L'_2(4)$, and Chang [3] showed that there are exactly three graphs with $m = 2$ in class (d_2), all three of them $S'_2(6) = T'(8)$. Now theorem 5.1 implies

5.3. Theorem (Seidel [11]). *The only strongly regular graphs with smallest eigenvalue -2 are the cocktailparty graphs (= complement of a set of disjoint edges), the triangular graphs $T(n+2) = S_2(n)$, the lattice graphs $L_2(n) = LS_2(n)$, the Shrikhande graph, the three Chang graphs, and the graphs of Peterson, Clebsch, and Schläfli.*

For $m = 3$ there are 64 exceptional parameter sets, 28 of which correspond to known SRGs. In two cases nonexistence of the graphs is known, and 34 cases are still undecided. A list will appear elsewhere.

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