On graphs whose spectral radius is bounded by $\frac{3}{2}\sqrt{2}$

by

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Abstract

The structure of graphs whose largest eigenvalue is bounded by $\frac{3}{2}\sqrt{2}$ (≈ 2.1312) is investigated. In particular, such a graph can have at most one circuit, and has a natural quipu structure.

Keywords: spectrum of graphs, largest eigenvalue, pendant trail, quipu

2000 MSC Classification: 05C50, 05C75

1 Introduction

Restrictions on the largest, second largest, or smallest eigenvalue of graphs force these to have a very special structure; in a number of cases, even a full classification of the resulting graphs is possible.

Cameron, Goethals, Seidel & Shult [5] characterized generalized line graphs as the graphs with smallest eigenvalue ≥ -2, apart from finitely many exceptions related to the exceptional root systems E_6, E_7, E_8. For this and smallest eigenvalues just below < -2, see chapter 3 of Brouwer, Cohen & Neumaier [2], Bussemaker & Neumaier [4], and Cvetković, Rowlinson & Simić [7]. For smallest eigenvalue $-1-\sqrt{2}$ and sufficiently large valency see Hoffman [13] and Woo & Neumaier [21]. Spielman [20] gives an application of smallest eigenvalue bounds to graph isomorphism testing.

Implications of bounds on the second largest eigenvalue can be found in [15, 8, 9, 10, 17, 18]; applications to expanders are in [1, 11].

The class of all graphs G whose largest eigenvalue $\lambda_{\max}(G)$ is bounded by 2 has been completely determined by Smith [19]. Later, with the help of tools developed by Hoffman [12], Cvetkovic et al. [6] gave a nearly complete description of all graphs G with $2 < \lambda_{\max}(G) \leq \sqrt{2+\sqrt{5}}$ (≈ 2.0582). Their description was completed by Brouwer & Neumaier [3] using the concept of exit values defined in Neumaier [15].

In this paper, we use exit values to examine the structure of graphs G with

$$\sqrt{2+\sqrt{5}} < \lambda_{\max}(G) \leq \frac{3}{2}\sqrt{2}(\approx 2.1312).$$

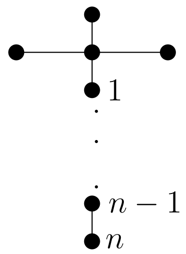
The resulting families of graphs resemble the knotted strings used by the Incas for information storage; we therefore use their term **quipus** for graphs with the following properties:

1.1 Definition.

(i) An **open quipu** is a tree G of maximum valency 3 such that all vertices of degree 3 lie on a path.

(ii) A **closed quipu** is a connected graph G of maximum valency 3 such that all vertices of degree 3 lie on a circuit, and no other circuit exists.

(iii) A **dagger** is a path with a 3-claw attached to an end vertex, i.e., one of the following graphs $T_0(n), n \geq 2$. (It has maximum valency 4 and hence is not a quipu in the present sense.)



$T_0(n)$

Our main result is the following theorem.

1.2 Theorem. *A graph G whose largest eigenvalue $\lambda_{\max}(G)$ satisfies*

$$2 < \lambda_{\max}(G) \leq \frac{3}{2}\sqrt{2}$$

is either an open quipu, a closed quipu, or a dagger.

We also show that, conversely, every dagger and all quipus with (in a precise sense) sufficiently long gaps between the vertices of degree 3 belong to the set $\mathcal{S}$ of connected graphs with largest eigenvalue > 2 and $\leq \frac{3}{2}\sqrt{2}$.

The set of minimal forbidden subgraphs for $\mathcal{S}$ is infinite, as it contains (at least) all graphs $T_2(n)$ ($n \geq 1$) defined below.

The result by SHEARER [16] that every number larger than $\sqrt{2+\sqrt{5}}$ is the limit of the largest eigenvalues of a sequence of graphs (and indeed of a class of open quipus called caterpillars) suggests that a complete description of the graphs in $\mathcal{S}$ should be rather intricate.

2 Preliminaries

2.1 Lemma. *If H is a (not necessarily induced) subgraph of G then $\lambda_{\max}(H) \leq \lambda_{\max}(G)$.*

Proof. This well-known result follows directly from the existence of a Perron eigenvector of H which may be applied to the Rayleigh quotient of the adjacency matrix of G. □

We say that an edge e lies on a *pendant trail* if there exists a path $v_1, v_2, ..., v_k$ in G such that v_1 is an end point of e, v_k is univalent, and all other vertices in the path are of degree two. HOFFMAN [12] introduced this concept to prove the following result.

2.2 Lemma. *(*Hoffman*)*

Let e be an edge of G and H formed from G by deleting e and replacing it with a path of length two. Then

(i) $\lambda_{\max}(H) > \lambda_{\max}(G)$ if e is on a pendant trail.

(ii) $\lambda_{\max}(G) \geq \lambda_{\max}(H)$ otherwise, with equality iff $G = C_n$ or $G = D_n$, where C_n is the n-gon, and D_n is the graph obtained from C_n by joining an extra vertex to a vertex of C_n. $\square$

In particular, the largest eigenvalue decreases if we make the side chains (pendant trails) of a quipu shorter, or if we lengthen the paths between two branching knots (degree 3 vertices) of a quipu.

In Neumaier [15], partial eigenvectors and exit values were introduced as follows: A vector e is a **λ-partial eigenvector** with respect to the vertex $z \in G$ iff $e(z) = 1$, and the relation

$$\sum_{x \sim y} e(x) = \lambda e(y)$$

holds for all $y \in G \setminus \{z\}$. In this case, the number $\epsilon_{G,z}(\lambda) = \lambda - \sum_{x\sim z} e(x)$ is called the **λ-exit value** of G with respect to z; if λ and G are given in the context, we simply write $\epsilon(z)$ for this exit value. (Here we use the convention of [2] to write $z \in G$ to express that z is a vertex of the graph G; a similar notation will be used later without comment.) Obviously, if the λ-exit value ϵ is zero, then λ is an eigenvalue of G, and e a corresponding eigenvector.

2.3 Theorem. *(=Theorem 2.4 in [15])*

Let e be a λ-partial eigenvector of G with respect to $z \in G$ such that λ is not an eigenvalue of $G \setminus \{z\}$, $e > 0$, and $\epsilon(z) > 0$. Then $\lambda_{\max}(G) < \lambda$. $\square$

In the following, let $\mathcal{S}$ be the set of connected graphs G with $2 < \lambda_{\max}(G) \leq \frac{3}{2}\sqrt{2}$. No G can attain $\lambda_{\max}(G) = \frac{3}{2}\sqrt{2}$, since this is not an algebraic integer. (Its minimal polynomial is $2\lambda^2 - 9$, hence cannot divide the characteristic polynomial of the graph, which has integer coefficients and highest coefficient 1.)

G always denotes a graph from $\mathcal{S}$, and we denote the maximum vertex degree of G by $d(G)$. We always take partial eigenvectors and exit values with respect to $\lambda = \lambda_0$, where

$$\lambda_0 = \frac{3}{2}\sqrt{2}.$$

Note that

$$\lambda_0 = \mu + \mu^{-1}, \text{ where } \mu = \sqrt{2}, \tag{1}$$

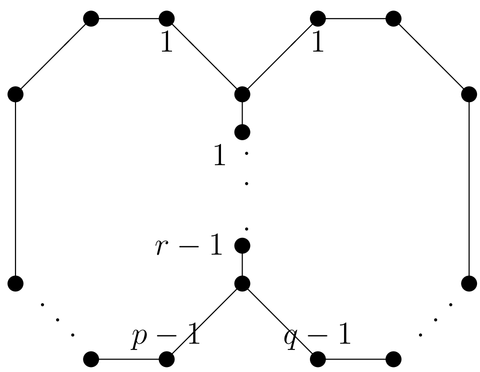
and we shall use these relations without reference.

2.4 Proposition. *Any graph $G \in \mathcal{S}$ has maximum degree $d(G) \in \{3, 4\}$.*

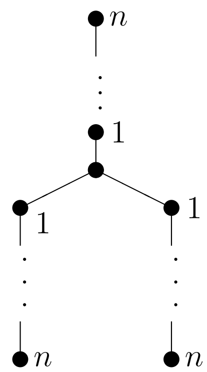
Proof. Suppose G has a vertex of degree five or more, then the 5-claw $K_{1,5}$ is a subgraph of G. Lemma 2.1 implies $\lambda_{\max}(G) \geq \lambda_{\max}(K_{1,5}) \geq 2.236 > \lambda_0$, contradiction. Hence $d(G) \leq 4$. On the other hand, $d(G) > 2$, for otherwise G would be either a path or a circuit, and so $\lambda_{\max}(G) \leq 2$, contradiction. □

3 Proof of the main result

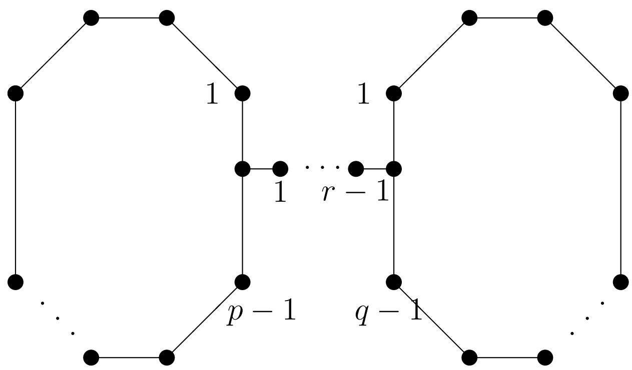
We need the following graphs:



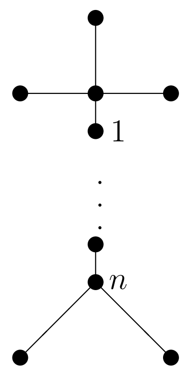
$Z_1(p, q, r), p, q \geq 2, r \geq 1$



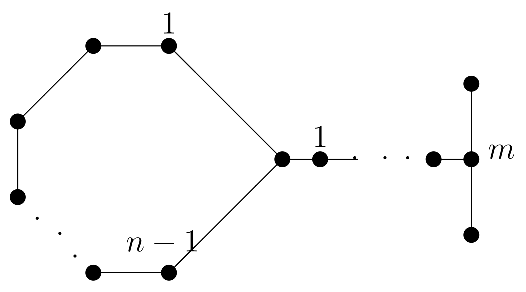
$T_1(n), n \geq 1$



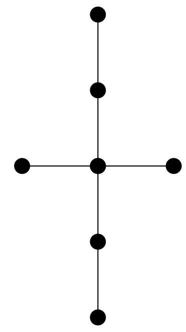
$Z_2(p, q, r), p, q \geq 3, r \geq 1$



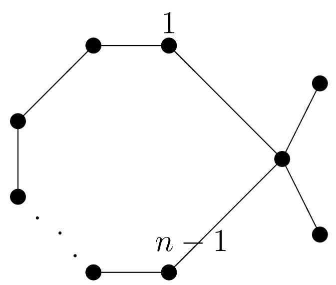
$T_2(n), n \geq 1$



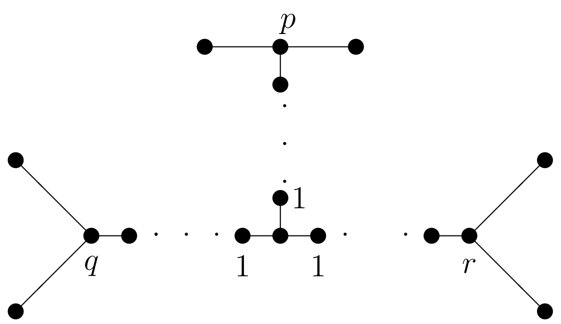
$Z_3(m,n), m \geq 1, n \geq 3$



T_3



$Z_4(n), n \geq 3$



$T_4(p,q,r), p,q,r \geq 1$

Note that, in particular, $Z_1(p,q,1)$ consists of two circuits with a single common edge, and $Z_2(p,q,1)$ consists of two circuits connected by an edge.

3.1 Lemma.
(i) $\lim_{n\to\infty} \lambda_{\max}(T_0(n)) = \lambda_0$.

(ii) $\lim_{n\to\infty} \lambda_{\max}(T_1(n)) = \lambda_0$.

Proof. This follows easily by solving the difference equation resulting from $Ax = \rho x$ for the Perron vector $x > 0$, noting that x has equal entries for vertices equivalent under a symmetry of the graph. For details, see Lemma 3.4 in HOFFMAN [12], which applies with

$$A_{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

to get (i), and with

$$A_{-1} = 0, \quad A_0 = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$$

to get (ii). Part (ii) also follows from LEPOVIĆ & GUTMAN [14]. □

3.2 Proposition. *A graph $G \in \mathcal{S}$ is either a tree or has exactly one circuit.*

Proof. Suppose G is not a tree, and that G has at least two distinct circuits. Then G must contain one of the graphs $Z_1(p,q,r)$ or $Z_2(p,q,r)$ as subgraph, for suitable $p, q \geq 2, r \geq 1$. (For example, two circuits with a single common vertex form a graph $Z_1(p,q,1)$.)

In the first case, Lemma 2.1 and Lemma 2.2 imply that, for $r > 1$,

$$\begin{aligned} \lambda_{\max}(G) &\geq \lambda_{\max}(Z_1(p,q,r)) > \lambda_{\max}(Z_1(m+1,m+1,m)) \\ &> \lambda_{\max}(T_1(m)) \end{aligned}$$

for all $m > \max\{p,q,r\}$, while for $r = 1$,

$$\begin{aligned} \lambda_{\max}(G) &\geq \lambda_{\max}(Z_1(p,q,r)) > \lambda_{\max}(Z_1(2m+2,m+1,1)) \\ &> \lambda_{\max}(T_1(m)) \end{aligned}$$

for all $m > \max\{p,q\}$. Similarly, in the second case,

$$\begin{aligned} \lambda_{\max}(G) &\geq \lambda_{\max}(Z_2(p,q,r)) > \lambda_{\max}(Z_2(m,m,m)) \\ &> \lambda_{\max}(Z_2(2m+1,m,m)) > \lambda_{\max}(T_1(m)) \end{aligned}$$

for all $m > \max\{p,q,r\}$. Thus, in both cases we have

$$\lambda_{\max}(G) \geq \lim_{m\to\infty} \lambda_{\max}(T_1(m)) = \lambda_0,$$

contradiction. □

3.3 Proposition. *The set of graphs in $\mathcal{S}$ with $d(G) = 4$ is precisely the set $\{T_0(n) \mid n \geq 2\}$; i.e., a graph with $d(G) = 4$ is in $\mathcal{S}$ iff it is a dagger.*

Proof. Let G be a graph with $d(G) = 4$.

Suppose first that G is not a tree. Then G contains a minimal circuit of size $n_0 \geq 3$, say. If this circuit contains a vertex of degree 4 then G contains a subgraph $Z_4(n_0)$; if not, G contains a subgraph $Z_3(m_0, n_0)$ for some $m_0 \geq 1$.

In the second case, Lemma 2.1 and Lemma 2.2 imply that

$$\begin{aligned} \lambda_{\max}(G) &\geq \lambda_{\max}(Z_3(m_0,n_0)) > \lambda_{\max}(Z_3(n,n)) \\ &> \lambda_{\max}(Z_3(n,2n+1)) > \lambda_{\max}(T_1(n)) \end{aligned}$$

for arbitrary $n >$max$\{m_0, n_0\}$. Similarly, in the first case,

$$\lambda_{\max}(G) \geq \lambda_{\max}(Z_4(n_0)) > \lambda_{\max}(Z_4(n+2)) > \lambda_{\max}(T_0(n))$$

for arbitrary $n > n_0$. Therefore, for either $j = 0$ or $j = 1$, we have

$$\lambda_{\max}(G) \geq \lim_{n\to\infty} \lambda_{\max}(T_j(n)) = \lambda_0,$$

contradiction.

Thus G must be a tree. Now G has at least one vertex z of degree 4. If G has another vertex with degree at least 3, then G must contain $T_2(n_0)$ as subgraph for some $n_0 \geq 1$, so

$$\lambda_{\max}(G) \geq \lambda_{\max}(T_2(n_0)) > \lambda_{\max}(T_2(n)) > \lambda_{\max}(T_0(n))$$

for all $n > n_0$, giving

$$\lambda_{\max}(G) \geq \lim_{n\to\infty} \lambda_{\max}(T_0(n)) = \lambda_0,$$

contradiction. Thus all vertices different from z have valency at most 2.

Since $\lambda_{\max}(T_3)(\approx 2.136) > \lambda_0$, T_3 cannot be a subgraph of G. It follows that $G = T_0(n)$ for some n.

Finally, since $\lambda_{\max}(T_0(1)) = 2$, $\lambda_{\max}(T_0(2))(\approx 2.074) > 2$, and $\lambda_{\max}(T_0(n))$ is monotonic increasing by Lemma 2.2 with limit λ_0 by Lemma 3.1, the subset of $\mathcal{S}$ of all graphs with at least a vertex of degree four is precisely $\{T_0(n) \mid n \geq 2\}$.
□

3.4 Proposition. *If $d(G) = 3$, then all the vertices of degree 3 lie on a path. Furthermore, if G is not a tree, then all vertices of degree 3 lie on the circuit.*

Proof. Let G be a counterexample to the proposition.

If G is a tree, it must have $T_4(p, q, r)$ as subgraph for some $p, q, r \geq 1$, and we find

$$\lambda_{\max}(G) \geq \lambda_{\max}(T_4(p, q, r)) > \lambda_{\max}(T_4(n, n, n)) > \lambda_{\max}(T_1(n))$$

for all $n >$max$\{p, q, r\}$. This implies

$$\lambda_{\max}(G) \geq \lim_{m\to\infty} \lambda_{\max}(T_1(n)) = \lambda_0,$$

contradiction.

If G has a circuit then, being a counterexample, not all vertices of degree 3 lie on the circuit, so that G has $Z_3(p, q)$ as subgraph for some $p \geq 1, q \geq 3$. This implies

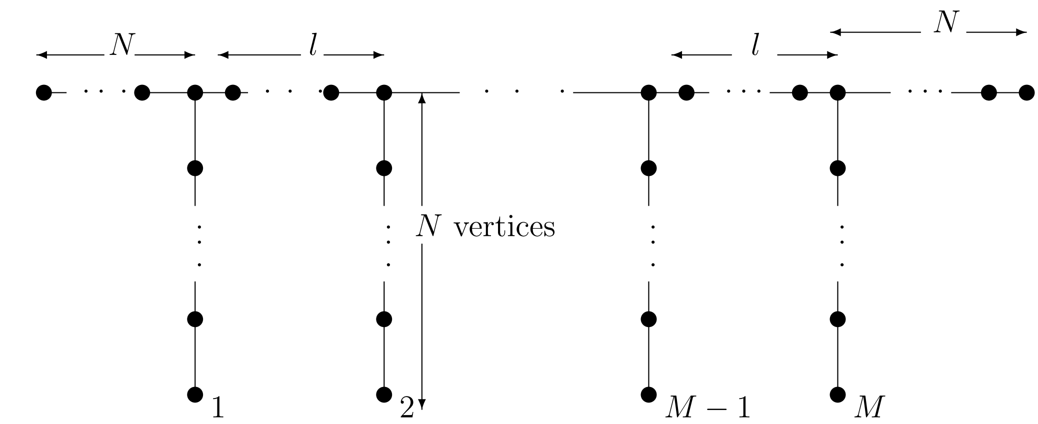
$$\lambda_{\max}(G) \geq \lambda_{\max}(Z_3(p, q)) > \lambda_{\max}(Z_3(2n+1, n)) > \lambda_{\max}(T_1(n))$$

for all $n > \max\{p, q\}$, contradiction. □

Clearly, the two propositions prove our main theorem stated in the introduction.

4 Open quipus with largest eigenvalue $< \frac{3}{2}\sqrt{2}$

For fixed positive integers M, N and another integer $l > 1$, we define the graph $G_{M,N,l}$ as the uniform open quipu with M branch points connected by chains of length l, having $M + 2$ side chains of length $N - 1$, cf. the following diagram.



In the following, we shall label the graph $G_{M,N,l}$ with labels from the set $C = C_0 \cup C_1 \cup C_2 \cup C_3$, where

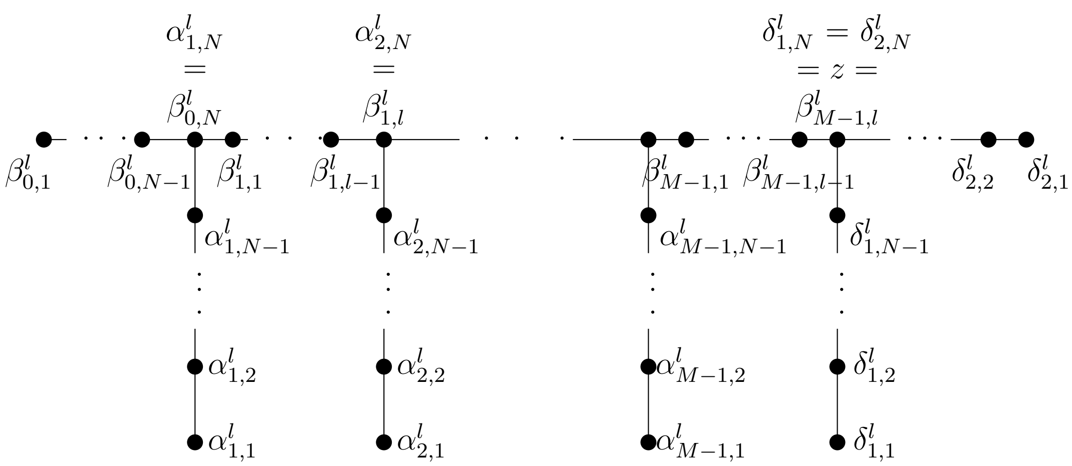
$$C_0 = \{\beta^l_{0,i} \mid i = 1, \ldots, N\},$$

$$C_1 = \{\beta^l_{m,i} \mid m = 1, \ldots, M-1; i = 1, \ldots, l\},$$

$$C_2 = \{\alpha^l_{m,n} \mid m = 1, \ldots, M-1; n = 1, \ldots, N\},$$

$$C_3 = \{\delta^l_{m,n} \mid m = 1, 2; n = 1, \ldots, N\},$$

as shown below. Note that vertices of degree three have more than one label.



Let z be the rightmost vertex of degree 3. For a vector e with coordinates indexed by the vertex set of $G_{M,N,l}$, the relation

$$\sum_{x \sim y} e(x) = \lambda_0 e(y) \tag{2}$$

holds for all $y \in G_{M,N,l} \setminus \{z\}$ iff the following relations (3) – (9) are satisfied:

$$\begin{aligned}
e(\alpha^l_{1,N}) &= e(\beta^l_{0,N}), && (3)\\
e(\alpha^l_{m,N}) &= e(\beta^l_{m-1,l}); \quad m = 2, ..., M-1, && (4)\\
e(\alpha^l_{m,n}) &= \mu^{N-n}\frac{\mu^{2n}-1}{\mu^{2N}-1} e(\alpha^l_{m,N}), \quad m = 1, ..., M-1; \quad n = 1, ..., N, && (5)\\
e(\delta^l_{m,n}) &= \mu^{N-n}\frac{\mu^{2n}-1}{\mu^{2N}-1} e(\delta^l_{m,N}), \quad m = 1, 2; \quad n = 1, ..., N, && (6)\\
e(\beta^l_{0,i}) &= \mu^{N-i}\frac{\mu^{2i}-1}{\mu^{2N}-1} e(\beta^l_{0,N}), \quad i = 1, ..., N, && (7)\\
e(\beta^l_{1,i}) &= \frac{\mu^{2(N+i)}-1}{\mu^i(\mu^{2N}-1)} e(\beta^l_{0,N}) - \frac{\mu^{2i}-1}{(\mu^2-1)\mu^{i-1}} e(\beta^l_{0,N-1}), \quad i = 1, ..., l, && (8)\\
e(\beta^l_{m+1,i}) &= \frac{\mu^{2(N+i)}-1}{\mu^i(\mu^{2N}-1)} e(\beta^l_{m,l}) - \frac{\mu^{2i}-1}{(\mu^2-1)\mu^{i-1}} e(\beta^l_{m,l-1}),\\
&\quad i = 1, ..., l; \quad m = 1, ..., M-2. && (9)
\end{aligned}$$

Indeed, (3) and (4) hold by definition; using (1), induction on n proves (5) and (6) apart from constant factors, which are chosen to match the case $n = N$. Induction on i similarly proves (7). Then induction on i proves (8) and (9).

Another induction argument now shows that a unique such vector e is determined by the normalization condition

$$e(\beta^l_{0,N}) = 1, \tag{10}$$

which we may impose without loss of generality.

4.1 Lemma.

$$\lim_{l\to\infty} \frac{e(\beta^l_{m,l-1})}{e(\beta^l_{m,l})} = \mu^{-1} \tag{11}$$

for $m = 1, \ldots, M-1$.

Proof. We first show that there are constants $C_{m,N}$ independent of i such that

$$\lim_{l\to\infty} \mu^{i-lm} e(\beta^l_{m,l-i}) = C_{m,N}. \tag{12}$$

(10) and (7) imply that $e(\beta^l_{0,N}) = 1$ and $e(\beta^l_{0,N-1}) =: q_N$ are independent of l. Hence (8) implies that

$$\mu^{i-l} e(\beta^l_{1,l-i}) = \frac{\mu^{2(N+l-i)}-1}{\mu^{2(l-i)}(\mu^{2N}-1)} e(\beta^l_{0,N}) - \frac{\mu^{2(l-i)}-1}{(\mu^2-1)\mu^{2(l-i)-1}} e(\beta^l_{0,N-1})$$

converges for $l \to \infty$ to

$$\frac{\mu^{2N}}{\mu^{2N}-1} - \frac{\mu}{\mu^2-1} q_N =: C_{1,N},$$

giving (12) for $m = 1$. If we assume that (12) holds for some $m \geq 1$, we find from (9) that

$$\mu^{i-l(m+1)} e(\beta^l_{m+1,l-i}) = \frac{\mu^{2(N+l-i)}-1}{\mu^{2(l-i)}(\mu^{2N}-1)} \mu^{-lm} e(\beta^l_{m,l}) - \frac{\mu^{2(l-i)}-1}{(\mu^2-1)\mu^{2(l-i)}} \mu^{1-lm} e(\beta^l_{m,l-1})$$

converges for $l \to \infty$ to

$$\frac{\mu^{2N}}{\mu^{2N}-1} C_{m,N} - \frac{1}{\mu^2-1} C_{m,N} =: C_{m+1,N},$$

giving (12) for $m+1$ in place of m. Thus (11) holds for $m+1$ in place of m, and by induction, (12) is proved. Taking quotients then proves the lemma. □

4.2 Lemma. *There is a number $L_{M,N}$ such that $e^l > 0$ (componentwise) for all $l > L_{M,N}$.*

Proof. (i) Clearly, the $e(\beta^l_{0,i})$ all have the same sign.

(ii) For $e(\beta^l_{k+1,1})$ to have the same sign as $e(\beta^l_{k,l})$, we check that $\frac{e(\beta^l_{k+1,l})}{e(\beta^l_{k,l})} > 0$. Indeed, we have

$$\lambda_0 e(\beta^l_{k,l}) = e(\beta^l_{k,l-1}) + e(\alpha^l_{k+1,N-1}) + e(\beta^l_{k+1,1}),$$

$$e(\beta^l_{k+1,1}) = \lambda_0 e(\beta^l_{k,l}) - e(\beta^l_{k,l-1}) - \mu \frac{\mu^{2(N-1)}-1}{\mu^{2N}-1} e(\beta^l_{k,l}),$$

$$e(\beta^l_{k+1,1}) = \left(\lambda_0 - \mu \frac{\mu^{2(N-1)}-1}{\mu^{2N}-1}\right) e(\beta^l_{k,l}) - e(\beta^l_{k,l-1}),$$

$$\frac{e(\beta^l_{k+1,1})}{e(\beta^l_{k,l})} = \left(\lambda_0 - \mu \frac{\mu^{2(N-1)}-1}{\mu^{2N}-1}\right) - \frac{e(\beta^l_{k,l-1})}{e(\beta^l_{k,l})} = \frac{\mu^{2(N+1)}-1}{\mu(\mu^{2N}-1)} - \frac{e(\beta^l_{k,l-1})}{e(\beta^l_{k,l})}.$$

Since $\lim_{l\to\infty} \frac{e(\beta^l_{k,l-1})}{e(\beta^l_{k,l})} = \mu^{-1}$, we have

$$\frac{\mu^{2(N+1)}-1}{\mu(\mu^{2N}-1)} > \frac{e(\beta^l_{k,l-1})}{e(\beta^l_{k,l})},$$

and hence $\frac{e(\beta^l_{k+1,1})}{e(\beta^l_{k,l})} > 0$, for l larger than a suitable number $l_{M,N,k}$.

(iii) For $e(\beta^l_{k+1,i})$ to have the same sign as $e(\beta^l_{k+1,1})$, we need to have $\dfrac{e(\beta^l_{k+1,1})}{e(\beta^l_{k+1,i})} > 0$. Using the abbreviation

$$\phi := \mu \frac{e(\beta^l_{k,l-1})}{e(\beta^l_{k,l})},$$

we obtain from (9) the relation

$$\frac{e(\beta^l_{k+1,1})}{e(\beta^l_{k+1,i})} = \mu^{i-1} \frac{(\mu^{2(N+1)}-1)(\mu^2-1)-(\mu^2-1)(\mu^{2N}-1)\phi}{(\mu^{2(N+i)}-1)(\mu^2-1)-(\mu^{2i}-1)(\mu^{2N}-1)\phi}.$$

Now a sufficient condition for this expression to be > 0 is that both $\dfrac{\mu^{2(N+1)}-1}{\mu^{2(N)}-1} > \phi$ and $\dfrac{(\mu^{2(N+i)}-1)}{\mu^{2(N+i)}+1-\mu^{2N}-\mu^{2i}} > \phi$. And since $\lim_{l\to\infty} \dfrac{e(\beta^l_{k,l-1})}{e(\beta^l_{k,l})} = \mu^{-1}$, this is indeed true whenever l is big enough, say $l > l'_{M,N,k}$.

Thus, with $L_{M,N} := \max\{l_{M,N,1}, ..., l_{M,N,M-2}, l'_{M,N,1}, ..., l'_{M,N,M-2}\}$, we have $e^l > 0$ for all $l > L_{M,N}$. □

4.3 Theorem. *For every $M, N, l \in \mathbb{N}^+$, there exist a number $L_{M,N,l}$ such that $l > \tilde{L}_{M,N,l}$ implies $G_{M,N,l} \in \mathcal{S}$.*

In particular, for every positive integer N, $\mathcal{S}$ contains open quipus with arbitrarily many side chains of length N.

Proof. The preceding lemma implies that for sufficiently large l, $e(z) \neq 0$. Because of (2), the vector $e(z)^{-1}e$ is the λ_0-partial eigenvector of $G_{M,N,l}$ at z, and the λ_0-exit-value of $G_{M,N,l}$ at z is given by

$$\begin{aligned} \epsilon^l &:= \lambda_0 - \left(\frac{e(\beta^l_{M-1,l-1})}{e(\beta^l_{M-1,l})} + \frac{e(\delta^l_{1,N-1})}{e(\beta^l_{M-1,l})} + \frac{e(\delta^l_{2,N-1})}{e(\beta^l_{M-1,l})}\right) \\ &= \lambda_0 - \left(\frac{e(\beta^l_{M-1,l-1})}{e(\beta^l_{M-1,l})} + \frac{e(\delta^l_{1,N-1})}{e(\delta^l_{1,N})} + \frac{e(\delta^l_{2,N-1})}{e(\delta^l_{2,N})}\right) \\ &= \lambda_0 - \left(\frac{e(\beta^l_{M-1,l-1})}{e(\beta^l_{M-1,l})} + \mu\frac{\mu^{2(N-1)}-1}{\mu^{2N}-1} + \mu\frac{\mu^{2(N-1)}-1}{\mu^{2N}-1}\right). \end{aligned}$$

By Lemma 4.1, the limit is

$$\lim_{l\to\infty} \epsilon^l = \lambda_0 - \left(\mu^{-1} + 2\mu\frac{\mu^{2(N-1)}-1}{\mu^{2N}-1}\right) = \mu - 2\mu\frac{\mu^{2(N-1)}-1}{\mu^{2N}-1} = \frac{\mu}{2^N-1} > 0.$$

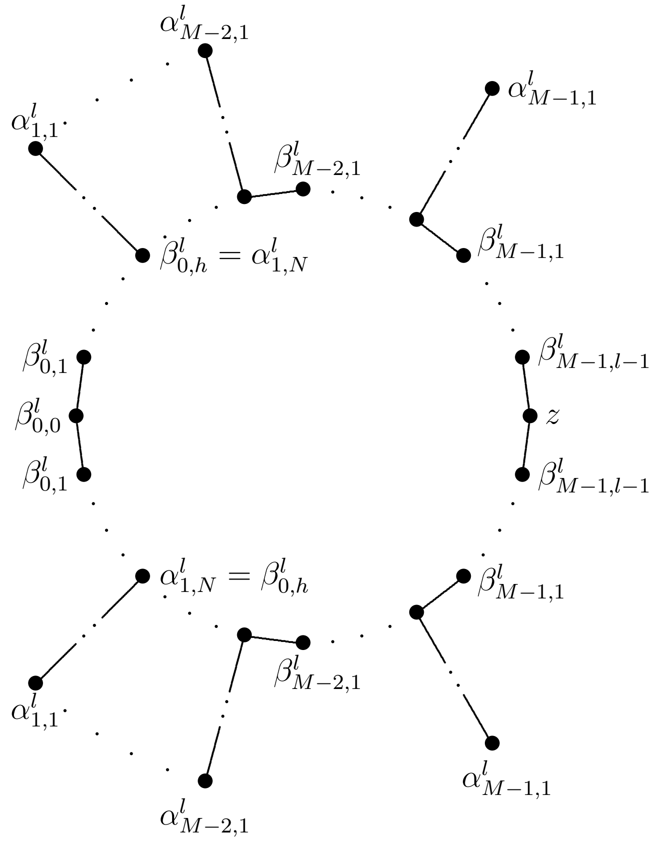
Thus Theorem 2.3 implies that, for sufficiently large l, $G_{M,N,l}$ has largest eigenvalue at most λ_0. □

Using Hoffman's Lemma 2.2, it is easy to relax the equal length restriction on gaps and side chains, resulting in a much larger family of open quipus in $\mathcal{S}$.

5 Closed quipus with largest eigenvalue $< \frac{3}{2}\sqrt{2}$

5.1 Theorem. *For any positive integer N, the set $\mathcal{S}$ contains closed quipus with arbitrarily many side chains of length N.*

Proof. Similar as before, we define the graph $Z_{M,N,l}$ and a labeling of its vertices by the following diagram. By abuse of notation, vertices symmetric with respect to the horizontal axis through the vertex z are given the same label since, by symmetry, their entries in e must be identical.



This graph is a closed quipu with an even number $2M-2$ of side chains of length $N-1$.

Now equations (3) to (6), (8) to (9) are still valid, and we have the following replacement of (7):

$$e(\beta^l_{0,i}) = \mu^{N-i}\frac{\mu^{2i}+1}{\mu^{2N}+1}e(\beta^l_{0,N}), \quad i \in \{1, ..., N\}. \tag{13}$$

Pleasantly, proceeding as before, we get the same values for $\lim_{l\to\infty} \frac{e(\beta^l_{m,l-1})}{e(\beta^1_{m,l})}$ as in the previous section. Since the limit of the λ_0-exit values,

$$\lim_{l\to\infty} \epsilon^l = \lambda_0 - 2\lim_{l\to\infty}\frac{e(\beta^l_{M-1,l-1})}{e(\beta^1_{M-1,l})} = \mu + \mu^{-1} - 2\mu^{-1} = \mu^{-1},$$

is positive, the old argument applies and shows that the graphs $Z_{M,N,l}$ belong to $\mathcal{S}$ when l is sufficiently large.

By attaching to $Z_{M,N,l}$ at z another side chain of length N, we get a closed quipu with an odd number $2M-1$ of side chains, and, again,

$$\begin{aligned}\lim_{l\to\infty}\epsilon^l &= \lambda_0 - \left(2\lim_{l\to\infty}\frac{e(\beta^l_{M-1,l-1})}{e(\beta^1_{M-1,l})} + \mu\frac{2^{N-1}-1}{2^N-1}\right)\\ &= \mu+\mu^{-1} - \left(2\mu^{-1}+\mu\frac{2^{N-1}-1}{2^N-1}\right) = \frac{\mu^{-1}}{2^N-1}\end{aligned}$$

is positive. Thus the same argument as before applies. □

Again, the fixed length restriction on gaps and side chains can be relaxed by Hoffman's lemma, giving a large class of closed quipus in $\mathcal{S}$.

Acknowledgment. We'd like to thank Waltraud Huyer for help with the figures.

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